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On an Alternative Model for the Optimal Production Planning Problem

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Abstract **– The paper considers an alternative economic hypothesis in relation to the one used in the optimal production planning model. Relevantthe conditions have a transparent economic meaning: the estimate of the total cost of resources for the production of a unit of a product should not exceed the price of this product. The results obtained are identical to known theorems of duality theory. The proofs given are of an elementary nature, only the simplest properties of the inequalities are used. Ways to increase the optimal amount of profit are considered. Specific standards for resource costs are indicated, with a decrease in which the optimal amount of profit can increase.**

Keywords **– Baseline, Evaluation, Duality, Resource, Optimization, Production Planning, Profit.**

I. Introduction

Application of mathematical methods to optimize production planningfirst studied by George B. Dantzig and L.W. Kantorowitsch [1-2]. Further development is given in [3-4]. Using a certain economic hypothesis, a dual problem was formulated and the main results of the theory of duality were obtained. At the same time, it is of interest to study the problem using an alternative economic hypothesis, which is the subject of this work.Consider the problem of production planning (see, for example, [5]):

$$\sum_{j=1}^{n} c_j x_j \quad = \max \qquad (1)$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{i}= 1, \dots, m \qquad (2)$$

$$x_j \geq 0 \text{j}= 1,\dots, \text{n} \qquad (3)$$

According to a well-known economic interpretation, the goal is to maximize the profit of a certain production at given prices, volumes of resources and norms for the costs of the latter for the production of a unit of production of each type. Here

b_i – stock of the i-th resource, i = 1, …, m;

a_{ij} - the need for the i-th resource for the production of a unit of the j-th product,

i = 1, …, m; j=1, …, n;

x_j is the planned volume of production of the j-th product, j = 1, …, n;

c_j - unit price of the j-th product, j = 1, …, n.

It is also assumed.

$$c_j > 0, \; b_i > 0, \quad a_{ij} \geq 0 \; (i = 1, \dots, m; j = 1, \dots, n) \qquad (4)$$

Let us assume that all supporting designs of the problem are non-degenerate.

We will call problem (1) - (3) the direct problem. An additional task can be attached to it, for which a certain economic hypothesis is used.

The commonly used hypothesis assumes that each resource is associated with its assessment, and for each product, the total assessment of the resources used in its manufacture should not be lower than the cost of the finished product:

$$\sum_{i=1}^{m} a_{ij}y_i \geq c_j \text{j}= 1, \dots, n \tag{5}$$

Here y_i is the estimate of the unit of the i-th resource. This estimate is not bounded from above and can take arbitrarily large values. In order to level this, a total estimate of resources to be minimized is introduced. The corresponding problem is called dual and has the form

$$\sum_{i=1}^{i=m} b_i\, y_i = \min \tag{6}$$

$$\sum_{i=1}^{i=m} a_{ij}y_i \geq c_j \quad j = 1, \dots, n \tag{7}$$

$$y_i \geq 0 \qquad i = 1, \dots, m \tag{8}$$

Certain results have been obtained for a dual pair of problems [3-4].

Instead of the dual problem, consider the following problem:

$$\sum_{i=1}^{i=m} b_i\, y_i = \max \tag{9}$$

$$\sum_{i=1}^{i=m} a_{ij}y_i \leq c_j \quad j = 1, \dots, n \tag{10}$$

$$y_i \geq 0 \qquad i = 1, \dots, m \tag{11}$$

We will call this task a companion one. A pair of problems (1-3) and (9-11) will be called a companion pair. Conditions (10) have a transparent economic meaning: the estimate of the total cost of resources for the production of a unit of product j should not exceed the price of this product. Consider jointly optimal solutions to problems (1-3) and (9-11).

Let us denote the optimal value of the objective function of problem (1) z_{max}:

$$\sum_{j=1}^{j=n} c_j x_j = z_{max} \tag{12}$$

Next, we denote the optimal value of the objective function of problem (9) v_{max}:

$$\sum_{i=1}^{i=m} b_i y_i = v_{max} \tag{13}$$

Multiply the i-th inequality of problem (2) by y_i and add up all m inequalities. Get

$$\sum_{i=1}^{i=m} y_i \sum_{j=1}^{j=n} a_{ij}\, x_j \leq \sum_{i=1}^{i=m} b_i\, y_i = v_{max} \tag{14}$$

$$\sum_{j=1}^{j=n} \left(\sum_{i=1}^{i=m} a_{ij} y_i\right) x_j \leq v_{max} \tag{15}$$

Subtract (12) from (15):

$$\sum_{j=1}^{j=n} \left(\sum_{i=1}^{i=m} a_{ij} y_i - c_j\right) x_j \leq v_{max} - z_{max} \tag{16}$$

It follows from (3) and (10) that the left side of inequality (16) is nonpositive. The maximum value of the left side is zero. We have:

$$v_{max} - z_{max} \geq 0 \quad (17)$$

$$v_{max} \geq z_{max} \quad (18)$$

Now let's multiply the j-th inequality of problem (10) by x_j and add up all n inequalities:

$$\sum_{j=1}^{j=n} x_j (\sum_{i=1}^{i=m} a_{ij} y_i) \leq \sum_{j=1}^{j=n} c_j x_j = z_{max} \quad (19)$$

$$\sum_{i=1}^{i=m} (\sum_{j=1}^{j=n} a_{ij} x_j) y_i \leq z_{max} \quad (20)$$

Subtract (13) from (20):

$$\sum_{i=1}^{i=m} (\sum_{j=1}^{j=n} a_{ij} x_j - b_i) y_i \leq z_{max} - v_{max} \quad (21)$$

It follows from (2) and (11) that the left side of inequality (21) is nonpositive. Its maximum value is zero. We have:

$$z_{max} - v_{max} \geq 0, \quad (22)$$

$$v_{max} \leq z_{max}$$ (23) From (18) and (23)

It follows that the optimal values of the objective functions of the problems coincide:

$$z_{max} = v_{max}$$ (24) From (16) and (24)

It follows:

$$\sum_{j=1}^{j=n} (\sum_{i=1}^{i=m} a_{ij} y_i - c_j) x_j \leq 0 \quad (25)$$

It follows from (3) and (10) that the left side of inequality (25) is nonpositive. Itsmaximumvalueiszero. Wherein

$$\sum_{j=1}^{j=n} (\sum_{i=1}^{i=m} a_{ij} y_i - c_j) x_j = 0 \quad (26)$$

All terms in (26) are nonpositive, i.e. each of them is zero:

$$(\sum_{i=1}^{i=m} a_{ij} y_i - c_j) x_j = 0 \qquad j = 1, \dots, \quad n \quad (27)$$

It follows from here that if $x_j > 0$, i.e. product j is included in the optimal production plan, then

$$\sum_{i=1}^{i=m} a_{ij} y_i - c_j = 0, \qquad j = 1, \dots, n \quad (28)$$

$$c_j = \sum_{i=1}^{i=m} a_{ij} y_i. \qquad J = 1, \dots, n \quad (29)$$

Thus, the total assessment of the resources used for its production is equal to its cost. If $c_j > \sum_{i=1}^{i=m} a_{ij} y_i$, then $x_j = 0$, i.e. product j is not included in the optimal production plan. The total estimate of the resources used for its production turns out to be less than its cost. Further, from (24) and (21) it follows:

$$\sum_{i=1}^{i=m} (\sum_{j=1}^{j=n} a_{ij} x_j - b_i) y_i \leq 0 \quad (30)$$

It follows from (2) and (11) that the left side of inequality (30) is nonpositive. Its maximum value is zero. Wherein.

$$\sum_{i=1}^{i=m} (\sum_{j=1}^{j=n} a_{ij} x_j - b_i)\, y_i = 0 \quad (31)$$

All terms in (31) are nonpositive, and therefore equal to zero:

$$(\sum_{j=1}^{j=n} a_{ij}x_j - b_i)y_i = 0, \quad i = 1, \ldots, m \tag{32}$$

Let $y_i > 0$. Then

$$b_i = \sum_{j=1}^{j=n} a_{ij}x_j \tag{33}$$

Thus, the i-th resource is completely used (scarce). An increase in a scarce resource can contribute to an increase in profits. If

$$\sum_{j=1}^{j=n} a_{ij}x_j < b_i, \tag{34}$$

then $y_i = 0$. Thus, the estimate of an underutilized (non-scarce) resource is equal to zero.

The results obtained are identical to known theorems of duality theory.

From the above, as well as the duality theorem on the coincidence of the optimal values of the objective functions of the primal and dual problems, it follows that the optimal values of the objective functions of the primal, concomitant, and dual problems coincide. Thus, using the dual or companion pair of problems is equivalent. The proofs given are of an elementary nature, only the simplest properties of the inequalities are used.

Let's consider some ways to increase the optimal amount of profit.Consider a related pair of problems. In the optimal solution of the direct problem, scarce resources are determined, the increase of which can lead to an increase in the optimal profit value. An increase in the optimal value of profit can also be facilitated by a decrease in certain standards of resource costs. Let $b_{i'}$ be one of the scarce resources, i.e.

$$\sum_{j=1}^{j=n} a_{i'j}x_j = b_{i'} \tag{35}$$

For any $x_j > 0$, when the cost standard $a_{i'j}$ decreases, the resource $b_{i'}$ ceases to be scarce. This can lead to an increase in the optimal profit margin. In this case, it is possible to use various strategies to reduce cost standards.

For some products, the prices in the optimal solution of the accompanying problem coincide with the total cost estimate of the resources used to manufacture these products. We call such prices marginal. Let $c_{j'}$ be one of the prices for which

$$\sum_{i=1}^{i=m} a_{ij'}y_i = c_{j'} \tag{36}$$

For any $y_i > 0$, as the cost norms $a_{ij'}$ decrease, the price $c_{j'}$ ceases to be a boundary value. This can help increase the optimal profit margin.

In both cases, it is necessary to re-solve the problems of the accompanying pair.

In the light of the foregoing, the case $a_{i'j'}$, may be of some interest, when the marginal price and the scarce resource simultaneously cease to be such. Let's call such elements key. They are located at the intersection of the columns of the original matrix, corresponding to the products included in the optimal production plan, and the rows corresponding to scarce resources. Let's call this submatrix the key submatrix. Reducing any element of this submatrix can increase the optimal profit. This information may be of some interest to designers, economists and other specialists. The growth of optimal profit can be facilitated by an increase in scarce equipment and marginal prices.

We will also be interested in other possibilities for increasing the optimal profit. Consider the direct problem (1-3). Consider hyperplanes:

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \quad i = 1, \ldots, m \tag{37}$$

We bring them to the following form:

$$\sum_{j=1}^{n} \frac{x_j}{(b_i/a_{ij})} = 1 \qquad a_{ij} \neq 0;\ i = 1,\ldots,m; \tag{38}$$

The values $\frac{b_i}{a_{ij}}$ (j = 1,…,n ; i=1,…,m) are segments cut off on the coordinate axes by hyperplanes (37). Denote

$$d_j = min_{i=1}^{i=m} \frac{b_i}{a_{ij}}, \qquad j = 1,\ldots,n. \tag{39}$$

Geometrically, d_j is the lowest of the points of intersection of the coordinate axis x_j with hyperplanes (37). Let's call points D_1 (d_1 , 0,…,0), D_2 (0, d_2 ,0,…,0), …, D_n (0,…,0, d_n) as axial boundary points. They are the vertices of the polyhedron of conditions (2)-(3), i.e. basic plans of the direct problem. They also belong to the hyperplane:

$$\sum_{j=1}^{j=n} \frac{x_j}{d_j} = 1. \tag{40}$$

The vertices of the solution polyhedron belonging to the coordinate axes form a subset of the axial boundary points. There are inequalities.

$$0 \leq x_j \leq d_j,\ j = 1, \ldots, n \tag{41}$$

Consider the optimal solution to problem (1), (2), (41) under the assumption that it is nondegenerate and unique.

Consider the case where the variable x_j in this solution takes on an upper bound value:

$$x_j = d_j. \tag{42}$$

$$d_j = min_{i=1}^{i=m} \frac{b_i}{a_{ij}} = \frac{b_{i'}}{a_{i'j}}. \tag{43}$$

With an increase in d_j, in principle, the objective function can also increase. An increase in d_j can occur in the following cases:

1. The stock of the resource $b_{i'}$ will increase;
2. The standard $a_{i'j}$ of resource consumptioni' per unit of product J will decrease;
3. The ratio $\frac{b_{i'}}{a_{i'j}}$ will increase.

Case 1 follows from the theory of duality, because the growth of the value of the total estimate of resources is associated with an increase in profit. Cases 2-3 provide further opportunities to increase profits. To check possibilities 1-3, it is necessary to resolve the original problem. Consider the accompanying problem.

$\sum_{i=1}^{i=m} b_i y_i$ = max (44)

$\sum_{i=1}^{i=m} a_{ij} y_i \leq c_j$, J=1, …, n (45)

$y_i \geq 0$, (46)

Here y_i is the estimate of resource unit i.Consider hyperplanes:

$\sum_{i=1}^{i=m} a_{ij} y_i = c_j$, j=1, ..., n (47)

We bring them to the following form:

$\sum_{i=1}^{i=m} \frac{y_i}{(c_j/a_{ij})} = 1, a_{ij} \neq 0.$ (48)

The values $\frac{c_j}{a_{ij}}$ are segments cut off on the coordinate axes by hyperplanes (47). Denote

$e_i = min_{j=1}^{j=n} \frac{c_j}{a_{ij}}$, i=1, …, m (49)

Geometrically, e_i the lowest point of intersection of the coordinate axis y_i with hyperplanes (47). Points $E_1(e_1, 0, \ldots, 0), E_2(0, e_2, 0, \ldots, 0), \ldots, E_m(0, \ldots, 0, e_m)$ are called axial boundary points. They are vertices of the polyhedral set defined by conditions (45)-(46) and belong to the hyperplane.

$\sum_{i=1}^{i=m} \frac{y_i}{e_i} = 1.$ (50)

There are inequalities,

$0 \leq y_i \leq e_i$, i = 1, …, m (51)

Consider the optimal solution to problem (44), (45), (51) under the assumption that it is nondegenerate and unique.

Let one of the variables in the optimal solution of the problem take its upper boundary value.

$y_i = e_i$, i = 1, …, m (52)

We have,

$e_i = min_{j=1}^{j=n} \frac{c_j}{a_{ij}} = \frac{c_{j'}}{a_{ij'}}.$ (53)

In principle, one can expect that with an increase in the upper bound e_i the value of the objective function (44) may also increase. In this case, the value of the objective function of the original problem will increase accordingly. An increase can occur in the following cases:

1. The price of the product $c_{j'}$ will increase;
2. The standard $a_{ij'}$ of the consumption of resource i per unit of product j' will decrease;
3. The ratio $\frac{c_{j'}}{a_{ij'}}$ will increase.

In case 1, an increase in the price of a product can lead to a corresponding increase in profits in the optimal solution. Cases 2-3 provide further opportunities to increase profits. To check possibilities 1-3, it is necessary to re-solve the original problem. A similar analysis for a dual pair of problems was carried out in the author's paper [6].

In the optimal solution of the problems of the dual pair, among other things, scarce resources are determined, the increase of which can lead to an increase in the optimal value of profit. The above applies to the accompanying pair of problems. At the same time, the prices of products are also determined, the increase of which can lead to an increase in the optimal profit value. Next, consider the following pair of problems.

$$\sum_{j=1}^{n} c_j x_j = \min \tag{54}$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad i = 1, \dots, m \tag{55}$$

$$x_j \geq 0 \quad j = 1, \dots, n \tag{56}$$

and

$$\sum_{i=1}^{i=m} b_i y_i = \min \tag{57}$$

$$\sum_{i=1}^{i=m} a_{ij} y_i \geq c_j \quad j = 1, \dots, n \tag{58}$$

$$y_i \geq 0 \quad i = 1, \dots, m \tag{59}$$

Here, thevalue$c_j$canbeinterpretedastheunitcostofproductj. Considerjointlyoptimalsolutionsofproblems (54-56) and (57-59).

Let us denote the optimal value of the objective function of problem (54)z_{min}:

$$\sum_{j=1}^{j=n} c_j x_j = z_{min} \tag{60}$$

Next, we denote the optimal value of the objective function of problem (57) v_{min}:

$$\sum_{i=1}^{i=m} b_i y_i = v_{min} \tag{61}$$

Multiply the i-th inequality of problem (55) by y_i and add up all m inequalities. Get

$$\sum_{i=1}^{i=m} y_i \sum_{j=1}^{j=n} a_{ij} x_j \geq \sum_{i=1}^{i=m} b_i y_i = v_{min} \tag{62}$$

$$\sum_{j=1}^{j=n} \left(\sum_{i=1}^{i=m} a_{ij} y_i\right) x_j \geq v_{min} \tag{63}$$

Subtract (60) from (63):

$$\sum_{j=1}^{j=n} \left(\sum_{i=1}^{i=m} a_{ij} y_i - c_j\right) x_j \geq v_{min} - z_{min} \tag{64}$$

It follows from (37) and (39) that the left side of inequality (64) is non-negative. Its minimum value is zero. Wherein

$$v_{min} - z_{min} \leq 0 \tag{65}$$

$$v_{min} \leq z_{min} \tag{66}$$

Now let's multiply the j-th inequality of problem (58) by x_j and add up all n inequalities:

$$\sum_{j=1}^{j=n} x_j \left(\sum_{i=1}^{i=m} a_{ij} y_i\right) \geq \sum_{j=1}^{j=n} c_j x_j = z_{min} \tag{67}$$

$\sum_{i=1}^{i=m}(\sum_{j=1}^{j=n} a_{ij}x_j)y_i \geq z_{min}$ (68)

Subtract (61) from (68):

$\sum_{i=1}^{i=m}(\sum_{j=1}^{j=n} a_{ij}x_j - b_i)y_i \geq z_{min} - v_{min}$ (69)

It follows from (55) and (59) that the left side of inequality (69) is non-negative. Its minimum value is zero. Wherein.

$z_{min} - v_{min} \leq 0, \qquad v_{min} \geq z_{min}$ (70)

From (66) and (70) it follows that the optimal values of the objective functions of the problems coincide:

$z_{min} = v_{min}$ (71)

From (64) and (71) it follows:

$\sum_{j=1}^{j=n}(\sum_{i=1}^{i=m} a_{ij}y_i - c_j)x_j \geq 0$ (72)

All terms in (72) are non-negative. The minimum value of the sum is zero, which is achieved when each term is equal to zero:

$(\sum_{i=1}^{i=m} a_{ij}y_i - c_j)\, x_j = 0$,. j = 1, …, n (73)

This implies that if $x_j > 0$, i.e. product j is included in the optimal production plan, then

$\sum_{i=1}^{i=m} a_{ij}y_i - c_j = 0$, (74)

$c_j = \sum_{i=1}^{i=m} a_{ij}y_i$. (75)

Thus, the total assessment of the resources used for its production is equal to its cost. If

$c_j < \sum_{i=1}^{i=m} a_{ij}y_i$, (76)

then $x_j = 0$, i.e. product j is not included in the optimal production plan. The total estimate of the resources used to produce it turns out to be greater than its value.

Further, from (69) and (71) it follows:

$\sum_{i=1}^{i=m}(\sum_{j=1}^{j=n} a_{ij}x_j - b_i)\, y_i \geq 0$ (77)

All terms in (77) are non-negative. The minimum value of the sum is equal to zero, which is achieved when each term is equal to zero. Wherein

$(\sum_{j=1}^{j=n} a_{ij}x_j - b_i)y_i = 0$, i=1, …, m (78)

Let $y_i > 0$. Then

$b_i = \sum_{j=1}^{j=n} a_{ij}x_j$ (79)

Thus, the i-th resource is used at the level of the lower bound, i.e. limited. Reducing the limit can help reduce the optimal value of the objective function. If

$\sum_{j=1}^{j=n} a_{ij}x_j - b_i > 0$, (80) Those. The total cost of the resource is above the limit level, then $y_i = 0$. Thus, the assessment of such a resource is equal to zero.

II. Conlusion

The results obtained are similar to the well-known theorems of duality theory.In this paper, we consider the dual and associated pairs of problems and show their equivalence. We note that the results are obtained within the framework of a uniform procedure, including: changing the order of summation; use of the simplest properties of inequalities, etc.

The proposed alternative model gives a symmetrical view of changing the parameters of the problem in order to increase the profit of the enterprise. This information may be of some interest to designers, economists and other specialists of the enterprise. A similar study was also carried out for the problem of minimizing the total cost of the production program.

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