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The Use of Boundary Values of Variables in Solving Problems of Optimal Production Planning

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Abstract **– In the problem of optimal production planning, a subset of the vertices of the polyhedron of the conditions of the problem lying on the coordinate axes is distinguished. Surfaces of the second order containing these vertices are considered. The point of tangency between the surface and the hyperplane parallel to the hyperplane of the objective function is taken as a solution to the problem. Also considered are issues related to obtaining an integer solution to the problem. Necessary and sufficient conditions for the variables to be integer are indicated. An algorithm for obtaining a suboptimal integer solution is proposed. The influence of the size of the boundary values of the variables in the optimal solution on the amount of profit is investigated. The factors of a possible increase in profit are indicated.**

Keywords **– Axial Boundary Point, Objective Function, Hyperplane, Tangency Point, Integer, Second Order Surface, Optimal Solution, Suboptimal Solution, Boundary Value, Polyhedral Set.**

I. Introduction

The problem of optimal production planning has historically been one of the first problems of linear programming [1-2]. In applied terms, it is characterized by a relatively low accuracy of the initial data, which makes it expedient to use approximate methods with expectedly lower costs.

Consider the problem of optimal production planning [3]:

$$f(x) = \sum_{j=1}^{n} c_j x_j \quad = max \tag{1}$$

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{i}= 1, \dots, m \tag{2}$$

$$x_j \ge 0 \qquad \text{j}= 1, \dots, \text{n} \tag{3}$$

According to the well-known economic interpretation, the goal is to maximize the profit of a certain production at given prices, volumes of resources and the norms of costs of the latter for the production of a unit of production of each type. It is also assumed that.

$$c_j > 0,\ \ b_i > 0,\ \ a_{ij} \ge 0\ \ (i = 1, \dots, m;\ \ j = 1, \dots, n) \tag{4}$$

Consider Hyperplanes:

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \qquad \text{i} = 1, \dots, \text{m} \tag{5}$$

Let's bring them to the following form:

$$\sum_{j=1}^{n} \frac{x_j}{(b_i/a_{ij})} = 1, \qquad a_{ij} \neq 0; \quad \text{I} = 1,\dots,\text{m} \tag{6}$$

The quantities $\frac{b_i}{a_{ij}}$ (j = 1,…,n; i = 1,…,m) are segments cut off on the coordinate axes by hyperplanes (6). We denote.

$d_j = min_{i=1}^{i=m} \frac{b_i}{a_{ij}}$, j = 1, …, n. (7)

Geometrically d_j is the lowest of the points of intersection of the coordinate axis x_j with hyperplanes (6). We will call the axial boundary points of the point.

D_1 (d_1 , 0, …, 0), D_2 (0, d_2 , 0, …,0), …, D_n (0, …,0, d_n). They are the vertices of the polyhedron of conditions (2) - (3), i.e. support plans of the problem, and belong to the hyperplane:

$$\sum_{j=1}^{j=n} \frac{x_j}{d_j} = 1. \quad (8)$$

Hyperplane (8) divides the space into two half-spaces.

The rest of the vertices of the polytope belong to the half-space that does not contain the point O (0, …, 0). The vertices of the solution polyhedron belonging to the coordinate axes form a subset of the axial boundary points.

Suitable surfaces can be used to estimate the optimal value of the objective function. We will be interested in second-order surfaces that have a common part with the first quadrant, located on the same side as the vertices of the polyhedron of the problem conditions, and also containing axial boundary points D_j, j = 1, …, n. Consider a tangent hyperplane to such a surface, parallel to the hyperplane of the objective function. The point of tangency is taken as an approximate solution to the problem. Its coordinates can be determined from the system of equations (9) and (10):

$$F(x_1, \ldots, x_n) = 0, \quad (9)$$

$$\frac{F'_{x_1}}{c_1} = \ldots = \frac{F'_{x_n}}{c_n}. \quad (10)$$

Here (9) is the equation of the corresponding surface of the second order. Next, consider the case when the surface equation is a level line of a quadratic form. Note that for a quadratic form $Q(x_1, x_2, \ldots, x_n)$ of the form

$$Q(x_1, x_2, \ldots, x_n) = \vec{x}^T A \vec{x}, \text{ где } A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} = a_{ji}, \quad i,j = 1, 2, \ldots, n; \quad (11)$$

$\vec{x}^T = (x_1 \ \ldots \ x_n)$, will be

$$(Q'_{x_1} \ \ldots \ Q'_{x_n})^T = 2A\vec{x} = 2\left(\sum_{j=1}^{j=n} a_{1j}x_j \ \ldots \ \sum_{j=1}^{j=n} a_{nj}x_j\right)^T \quad (12)$$

Equations (10) take the form:

$$\frac{\sum_{j=1}^{j=n} a_{1j}x_j}{c_1} = \ldots = \frac{\sum_{j=1}^{j=n} a_{nj}x_j}{c_n}. \quad (13)$$

Example

Let an ellipsoid be given

$$\left(\frac{x_1}{d_1}\right)^2 + \ldots + \left(\frac{x_n}{d_n}\right)^2 = 1 \quad (14)$$

Consider the tangent hyperplane to it, parallel to the hyperplane of the objective function. The point of tangency belonging to the first quadrant is taken as an approximate solution to the problem. Further,

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$$\frac{x_1}{d_1^2 c_1} = \dots = \frac{x_n}{d_n^2 c_n} = \mathrm{e} \tag{15}$$

We have:

$$x_j = \mathrm{e}c_j d_j^2 \Rightarrow \sum_{j=1}^{j=n} (ec_j d_j)^2 = 1, \Rightarrow e = \frac{1}{\sqrt{\sum_{j=1}^{j=n} (c_j d_j)^2}} \tag{16}$$

$$x_j = \frac{c_j d_j^2}{\sqrt{\sum_{j=1}^{j=n} (c_j d_j)^2}} \quad \mathrm{j} = 1, \dots, \mathrm{n} \tag{17}$$

Objective function value.

$$\sum_{j=1}^{j=n} c_j x_j = \sum_{j=1}^{j=n} \frac{(c_j d_j)^2}{\sqrt{\sum_{j=1}^{j=n} (c_j d_j)^2}} = \frac{\sum_{j=1}^{j=n} (c_j d_j)^2}{\sqrt{\sum_{j=1}^{j=n} (c_j d_j)^2}} \tag{18}$$

$$\sum_{j=1}^{j=n} c_j x_j = \sqrt{\sum_{j=1}^{j=n} (c_j d_j)^2} \tag{19}$$

The considered approach can be used, in particular, to construct an iterative scheme for approximating the optimal solution [4]. We also note the possibility of using different surfaces of the second order. For small problems, an approximate solution can be obtained using a calculator.

Next, we consider the properties of the optimal solution to the problem taking into account the boundary values of the variables.

Manufactured products are taken into account, in particular, according to such quantitative indicators as volume, weight, as well as by the piece (single, small-scale production). Accordingly, we will be interested in the optimal solution to problem (1) - (3), as well as the optimal solution to this problem under the condition that all variables are integer, i.e. optimal integer solution.

There are inequalities,

$$0 \leq x_j \leq d_j \qquad \mathrm{j} = 1, \dots, n \tag{20}$$

Consider the optimal solution to problem (1), (2), (20) under the assumption of its non-degeneracy and uniqueness. Let k resource constraints (2) become equal. Then n - k variables in (20) will take boundary values, and k variables will take intermediate values. Thus, the number of bottlenecks, i.e. the number of fully used resources is exactly equal to the number of variables that took intermediate values. For k = 0, all variables in (20) take boundary values. In this case, none of the resources (2) will be fully used. Conversely, if none of the resources is fully used, then all variables will take boundary values.

Further,

$$0 \leq x_j \leq [d_j] \qquad \mathrm{j} = 1, \dots, \mathrm{n} \tag{21}$$

Here $[d_j]$ is the integer part of the number d_j. Consider problem (1), (2), (21). If in the optimal solution k constraints (2) become equal, then the n-k variables in (21) will be boundary variables, i.e. integer values, and k variables are intermediate values. For k = 0, all variables in (21) will take integer values. In this case, none of the resources will be fully used. Conversely, if none of the resources are fully used, then all variables will take integer values. This solution is at least a suboptimal integer solution to the original problem (1) - (3).

Currently, there are no effective algorithms for obtaining an optimal integer solution to a linear programming problem [5-6]. It seems expedient to use the potential of the simplex method to obtain an integer solution of the problem under consideration in a reasonable time. From the practice of using the simplex method, it is known that the number of steps to obtain the optimal solution is approximately equal to O (1) (m + n), where m is the number of equations, n is the number of variables, O (1) is about 4 (see [7]).

When solving problems (1), (2), (21) by the simplex method, some of the variables in the optimal solution take boundary values, i.e. integer values. Some variables that take intermediate values can also be integer values. Let the first l variables in the optimal solution take integer values. Accordingly, n - l variables will be non-integer. Consider the problem

$$f(x) = \sum_{j=l+1}^{j=n} c_j x_j \ = max \tag{22}$$

$$\sum_{j=l+1}^{j=n} a_{ij} x_j \le b_i' \quad i = 1, \dots, m \tag{23}$$

$$x_j \ge 0 \qquad j = l+1, \dots, n \tag{24}$$

Here

$$b_i' = b_i - \sum_{j=1}^{j=l} a_{ij} x_j \qquad i = 1, \dots, m \tag{25}$$

In accordance with the above algorithm, we obtain the inequalities,

$$0 \le x_j \le d_j' \qquad j = l+1, \dots, n \tag{26}$$

$$0 \le x_j \le [d_j'] \qquad j = l+1, \dots, n \tag{27}$$

Next, using the simplex method, we solve problem (22), (23), (27).

The procedure can be used as many times as necessary to obtain fully integer values of all unknowns. The number of repetitions of the procedure does not exceed n. The resulting solution is a suboptimal integer solution to the original problem.

The corresponding value of the objective function will be denoted by z_smax. The optimal value of the objective function of the integer problem (1) - (3) belongs to the interval.

(z_{smax}, z_{max}), where z_{max} is the optimal value of the objective function of problem (1) - (3).

The described technique can also be used if some of the variables of the original problem (1) - (3) in their meaning take non-integer values.

Next, we investigate the boundary values of the variables in the optimal solution (see also [8]). Consider the case when the variable x_j in this solution takes the upper boundary value:

$$x_j = d_j. \tag{28}$$

We have

$$d_j = min_{i=1}^{i=m} \frac{b_i}{a_{ij}} = \frac{b_{i'}}{a_{i'j}}. \tag{29}$$

With increasing d_j, in principle, the objective function can also increase. An increase in d_j can occur in the f-

-ollowing cases:

1. The stock of the resource $b_{i'}$ will increase;
2. The rate $a_{i'j}$ consumption of the resource i' per unit of product J will decrease;
3. The ratio $\frac{b_{i'}}{a_{i'j}}$ will increase.

Case 1 follows from the theory of duality, because an increase in the value of the total estimate of resources is associated with an increase in profits. Cases 2-3 present further opportunities to increase profits. To test possibilities 1-3, you need to re-solve the original problem.

Consider the dual problem to (1) - (3):

$$\sum_{i=1}^{i=m} b_i y_i = \min \tag{30}$$

$$\sum_{i=1}^{i=m} a_{ij} y_i \geq c_j, \quad J=1, \ldots, m \tag{31}$$

$$y_i \geq 0, \tag{32}$$

Here y_i is the estimate of the unit of resource i.

Consider hyperplanes:

$$\sum_{i=1}^{i=m} a_{ij} y_i = c_j, \quad j = 1, ..., n \tag{33}$$

Let's bring them to the following form:

$$\sum_{i=1}^{i=m} \frac{y_i}{(c_j/a_{ij})} = 1, a_{ij} \neq 0. \tag{34}$$

The quantities $\frac{c_j}{a_{ij}}$ are segments cut off on the coordinate axes by hyperplanes (34). We denote

$$e_i = max_{j=1}^{j=n} \frac{c_j}{a_{ij}}, i=1, \ldots, m \tag{35}$$

Geometrically, e_i is the highest point of intersection of the y_i coordinate axis with hyperplanes (34). Similar to the pointwill be called axial boundary points.

$E_1(e_1, 0, \ldots, 0)$, $E_2(0, e_2, 0, \ldots, 0)$, ... , $E_m(0, \ldots, 0, e_m)$. They are the vertices of the polyhedral set defined by conditions (31) - (32) and belong to the hyperplane (31)-(32).

$$\sum_{i=1}^{i=m} \frac{y_i}{e_i} = 1. \tag{36}$$

Hyperplane (36) divides the space into two half-spaces. The rest of the vertices of the polyhedral set belong to the half-space not containing the point O (0, …, 0). There are inequalities,

$$y_i \geq e_i, \qquad i = 1, \ldots, m \tag{37}$$

Consider the optimal solution to problem (30), (31), (37) under the assumption of its non-degeneracy and uniqueness.

Let one of the variables in the optimal solution of the problem take its lower boundary value.

$$y_i = e_i, \quad i = 1, \ldots, m \tag{38}$$

We have

$$e_i = max_{j=1}^{j=n} \frac{c_j}{a_{ij}} = \frac{c_{j'}}{a_{ij'}}. \quad (39)$$

In principle, it can be expected that with an increase in the lower bound e_i, the value of the target function can also increase (30). In this case, the value of the objective function of the original problem will correspondingly increase (according to the duality theorem).

An increase can occur in the following cases:

1. Product prise $c_{j'}$ will increase;

2. The rate $a_{ij'}$ consumption of the resource i per unit of product j' will decrease;

3. The ratio $\frac{c_{j'}}{a_{ij'}}$ will increase.

In case 1, an increase in the price of a product can lead to a corresponding increase in profit in the optimal solution. Cases 2-3 present further opportunities to increase profits. To test possibilities 1-3, it is necessary to re-solve the original problem.

The conducted research indicates the fundamental possibility of increasing profits with directional variation of the parameters immanent to the boundary values of the variables. To test these possibilities, the original problem is re-solved. This can serve as a definite addition to the well-known postoptimal analysis.

Thus, in this work, the following results were obtained for the problem of optimal production planning:

- Formulas are given for the approximate solution of the problem;

- The necessary and sufficient conditions for the integrality of the solution are indicated:

- An algorithm for obtaining a suboptimal integer solution in a finite number of steps is proposed;

- The factors of a possible increase in profit have been investigated.

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A. Shapiro, Mathematician by profession (State University, Kharkiv, 1961), he worked at the company „Novo-Kramatorsk Machine-Building Plant“ as head of the programming department, later as an assistant professor at engineering academy, Kramatorsk. He has PhD degree in Economics (Central Economics and Mathematics Institute of the USSR Academy of Sciences, Moscow, 1979). He has published books: „Why should we solve problems?“ („Prosveshenie“, Moskow, 1996), „Stories with Science“, „The Salt of Mathematics“ („Altaspera“, Canada, 2020). Sinse 2000 he lives in Germany.