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On the Complete Elliptic Integrals and Babylonian Identity VIII: An Approximation for the Complete Elliptic Integral of Second Kind

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***Abstract* – We demonstrated an approximation for the complete elliptic integral of the second kind with Bessel functions, using formula proved in previous paper.**

I. INTRODUCTION

By means of the complete elliptic integral of the second kind, wedemonstrated the following approximation formula:

$$E(k) \cong \pi\left\{\frac{3k^2}{8}+\frac{85k^4}{1024}-\frac{1}{2}+\left(1-\frac{91k^2}{96}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2+\frac{(29-4k^2)k}{24}I_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right)+\frac{(29-k^2)k^2}{48}\left[I_1\left(\frac{k}{2}\right)\right]^2\right\},$$

for$0<k\le 1/2$.

II. THEOREM

Theorem 1. We have

$$E(k) \cong \pi\left\{\frac{3k^2}{8}+\frac{85k^4}{1024}-\frac{1}{2}+\left(1-\frac{91k^2}{96}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2+\frac{(29-4k^2)k}{24}I_0\left(\frac{k}{2}\right)I_1\left(\frac{k}{2}\right)+\frac{(29-k^2)k^2}{48}\left[I_1\left(\frac{k}{2}\right)\right]^2\right\},$$

where $E(k)$ is the complete elliptic integral of second kind.

Proof. In previous paper [1], we demonstrated that

$$(1)\ \frac{2E(k)}{\pi}=\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}\binom{2n}{n}\left(\frac{k}{2}\right)^{2n},$$

in other word,

$$(2)\ \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}=\frac{2E(k)}{\pi}.$$

we calculate

$$(3)\ \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}={}_1F_2\left(\left.\begin{matrix}-\frac{1}{2}\\1,1\end{matrix}\right|\frac{k^2}{4}\right)=\left(1-\frac{k^2}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2+\frac{k^2}{2}\left[I_1\left(\frac{k}{2}\right)\right]^2+kI_1\left(\frac{k}{2}\right)I_0\left(\frac{k}{2}\right).$$

we subtract the left-hand side of (2) for the left-hand side of (3), and encounter

$$(4)\ \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!-1}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}.$$

On the other hand, we note that

$$(5)\ (2n)!-1\ge(2n-1)!\,n+\frac{4}{3}n^5+\frac{5}{4}n^3+\frac{5}{12}n,$$

for$n\ge 3$.From (4) and (5), we have

$$(6)\ \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}>\sum_{n=0}^{2}(-1)^n\frac{(2n)!-1}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}$$
$$+\sum_{n=3}^{\infty}(-1)^n\frac{1}{(n!)^2}\left[(2n-1)!\,n+\frac{4}{3}n^5+\frac{5}{4}n^3+\frac{5}{12}n\right]\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}$$

$$= -\frac{23k^4}{512} - \frac{k^2}{8} + \sum_{n=3}^{\infty}(-1)^n \frac{(2n-1)!\,n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} + \frac{4}{3}\sum_{n=3}^{\infty}(-1)^n \frac{n^5}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} + \frac{5}{4}\sum_{n=3}^{\infty}(-1)^n \frac{n^3}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} + \frac{5}{12}\sum_{n=3}^{\infty}(-1)^n \frac{n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}$$

$$= -\frac{23k^4}{512} - \frac{k^2}{8} + \sum_{n=3}^{\infty}(-1)^n \binom{2n-1}{n}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} + \frac{4}{3}\sum_{n=3}^{\infty}(-1)^n \frac{n^5}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} + \frac{5}{4}\sum_{n=3}^{\infty}(-1)^n \frac{n^3}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} + \frac{5}{12}\sum_{n=3}^{\infty}(-1)^n \frac{n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n}$$

we calculate

$$(7)\ \sum_{n=3}^{\infty}(-1)^n \binom{2n-1}{n}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} = \frac{3k^4}{128} + \frac{k^2}{8} + \frac{E(k)}{\pi} - \frac{1}{2},$$

$$(8)\ \sum_{n=3}^{\infty}(-1)^n \frac{n^5}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} = -\frac{k^2}{8}\,{}_3F_4\left(\left.\begin{matrix}\frac{1}{2},2,2\\1,1,1,1\end{matrix}\right|\frac{k^2}{4}\right) + \frac{k^4}{16} + \frac{k^2}{8}$$
$$= \frac{k^4}{16} + \frac{k^2}{8} - \frac{k^4}{64}\left[I_1\left(\frac{k}{2}\right)\right]^2 - \frac{k^3}{8} I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) - \frac{k^2}{16}\left(2 + \frac{k^2}{4}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2,$$

$$(9)\ \sum_{n=3}^{\infty}(-1)^n \frac{n^3}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} = \frac{k^4}{64} + \frac{k^2}{8} - \frac{k^2}{8}\left[I_0\left(\frac{k}{2}\right)\right]^2$$

and

$$(10)\ \sum_{n=3}^{\infty}(-1)^n \frac{n}{(n!)^2}\binom{1/2}{n}\left(\frac{k}{2}\right)^{2n} = \frac{k^4}{256} + \frac{k^2}{8} - \frac{k^2}{8}\,{}_1F_2\left(\left.\begin{matrix}\frac{1}{2}\\2,2\end{matrix}\right|\frac{k^2}{4}\right)$$
$$= \frac{k^4}{256} + \frac{k^2}{8} - \frac{k^2}{4}\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{k^2}{4}\left[I_1\left(\frac{k}{2}\right)\right]^2 + \frac{k}{2} I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right).$$

From (2), (3), (6), (7), (8), (9) and (10), we deduced that

$$\frac{2E(k)}{\pi} - \left\{\left(1 - \frac{k^2}{2}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{k^2}{2}\left[I_1\left(\frac{k}{2}\right)\right]^2 + kI_1\left(\frac{k}{2}\right) I_0\left(\frac{k}{2}\right)\right\} > -\frac{23k^4}{512} - \frac{k^2}{8} + \frac{3k^4}{128} + \frac{k^2}{8} + \frac{E(k)}{\pi} - \frac{1}{2}$$
$$+ \frac{4}{3}\left\{\frac{k^4}{16} + \frac{k^2}{8} - \frac{k^4}{64}\left[I_1\left(\frac{k}{2}\right)\right]^2 - \frac{k^3}{8} I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) - \frac{k^2}{16}\left(2 + \frac{k^2}{4}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2\right\} + \frac{5}{4}\left\{\frac{k^4}{64} + \frac{k^2}{8} - \frac{k^2}{8}\left[I_0\left(\frac{k}{2}\right)\right]^2\right\}$$
$$+ \frac{5}{12}\left\{\frac{k^4}{256} + \frac{k^2}{8} - \frac{k^2}{4}\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{k^2}{4}\left[I_1\left(\frac{k}{2}\right)\right]^2 + \frac{k}{2} I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right)\right\},$$

ergo,

$$\frac{E(k)}{\pi} > \frac{3k^2}{8} + \frac{85k^4}{1024} - \frac{1}{2} + \left(1 - \frac{91k^2}{96}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{(29 - 4k^2)k}{24} I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) + \frac{(29 - k^2)k^2}{48}\left[I_1\left(\frac{k}{2}\right)\right]^2,$$

since $0 < k \le 1/2$, then we may assume easily that

$$E(k) \cong \pi\left\{\frac{3k^2}{8} + \frac{85k^4}{1024} - \frac{1}{2} + \left(1 - \frac{91k^2}{96}\right)\left[I_0\left(\frac{k}{2}\right)\right]^2 + \frac{(29 - 4k^2)k}{24} I_0\left(\frac{k}{2}\right) I_1\left(\frac{k}{2}\right) + \frac{(29 - k^2)k^2}{48}\left[I_1\left(\frac{k}{2}\right)\right]^2\right\}. \square$$

REFERENCES

[1] Read our previous article "*On the complete elliptic integrals and Babylonian Identity VI: the complete elliptic integral of second kind*", 2014.

Table 1: In this table, we have: first column: m; second column: $k = 1/m$; third column: $\pi\left[\frac{85}{1024m^4} + \frac{3}{8m^2} +\right.$ *1−9196m2I012m2+29m2−148m4I112m2+29m2−424m3I112mI012m−12*; fourth column: *E1m,* fifth column: $E\left(\frac{1}{m}\right)/\pi\left[\frac{85}{1024m^4} + \frac{3}{8m^2} + \left(1 - \frac{91}{96m^2}\right)\left[I_0\left(\frac{1}{2m}\right)\right]^2 + \frac{29m^2-1}{48m^4}\left[I_1\left(\frac{1}{2m}\right)\right]^2 + \frac{29m^2-4}{24m^3} I_1\left(\frac{1}{2m}\right) I_0\left(\frac{1}{2m}\right) - \frac{1}{2}\right]$.

2	$\frac{1}{2}$	1.4548756512939791961	1.4674622093394271554	1.0086512947235410558
3	$\frac{1}{3}$	1.5196573696849274648	1.5262092342121874283	1.0043114090438809703
4	$\frac{1}{4}$	1.5420925038041505999	1.5459572561054650349	1.002506174105496636
5	$\frac{1}{5}$	1.5524427231421820988	1.5549685462425292834	1.0016269992204510172
6	$\frac{1}{6}$	1.5580568513766824749	1.5598305371286827833	1.0011383960414750582
7	$\frac{1}{7}$	1.5614393391694568738	1.5627511292627544183	1.0008401159496822316
8	$\frac{1}{8}$	1.5636336677061683758	1.5646423092625568944	1.0006450625726601189
9	$\frac{1}{9}$	1.5651376302088243507	1.5659369093229590304	1.0005106765684421234
10	$\frac{1}{10}$	1.5662131783720514046	1.5668619420216682912	1.0004142243588393391
11	$\frac{1}{11}$	1.5670088415144249769	1.5675458333404971662	1.0003426858941990507
12	$\frac{1}{12}$	1.5676139402628208592	1.568065688643469198	1.0002881757868092192
13	$\frac{1}{13}$	1.5680848087518254294	1.568470079184143107	1.0002456948949236665
14	$\frac{1}{14}$	1.5684584032931590188	1.5687908392881918108	1.0002119507883248955
15	$\frac{1}{15}$	1.5687597841159573006	1.5690495404018847541	1.000184704050206603
16	$\frac{1}{16}$	1.5690064315069514698	1.5692612206533624497	1.000162388847677487
17	$\frac{1}{17}$	1.5692108395278960233	1.5694366235819572228	1.0001438838226029127
18	$\frac{1}{18}$	1.5693821303502344743	1.5695835902798978113	1.0001283689458209758
19	$\frac{1}{19}$	1.5695270901477335966	1.5697079520534112774	1.0001152333762271392
20	$\frac{1}{20}$	1.5696508518503276635	1.5698141184163880256	1.0001040145748768913
21	$\frac{1}{21}$	1.5697573559345742701	1.5699054737256423517	1.0000943571250092329
22	$\frac{1}{22}$	1.5698496680488761372	1.5699846504660967064	1.0000859842951639663
23	$\frac{1}{23}$	1.5699302021420589736	1.5700537211604400414	1.000078678031808379
24	$\frac{1}{24}$	1.5700008799117849128	1.5701143354650531983	1.0000722646431094104
25	$\frac{1}{25}$	1.5700632465274273294	1.5701678196428456723	1.0000666043967653096
26	$\frac{1}{26}$	1.5701185558133157843	1.5702152497673788669	1.0000615838553751729
27	$\frac{1}{27}$	1.5701678337677570053	1.570257506297885084	1.000057110156124458
28	$\frac{1}{28}$	1.570211926493411572	1.5702953152530112401	1.0000531066910094686
29	$\frac{1}{29}$	1.5702515367628382819	1.5703292796166320983	1.0000495098090807079
30	$\frac{1}{30}$	1.5702872521977722559	1.5703599035372227177	1.000046266273478804
31	$\frac{1}{31}$	1.5703195671903999749	1.5703876111506035821	1.0000433312821321782
32	$\frac{1}{32}$	1.5703489001059926282	1.5704127613492695165	1.0000406669137492807