Two-Scale Convergence and Time-Harmonic Maxwell's Equations

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Date of publication (dd/mm/yyyy): 13/08/2020

Abstract **– The paper is an application of two-scale convergence given by Nuestseng in 1989. We present the Maxwell's equations given in time-harmonic in R^3. At the end of studies, we obtained the homogenized problem and we prove a corrector result.**

Keywords **– Homogenization, Two-Scale Method, Weak Convergence, Periodic Extension, Microscopic Scale, Local Problem, Macroscopic Effects.**

I. Introduction

The object of this paper is to analyze the homogenization of the Maxwell's equations given in special harmonic case, by using two-scale convergence, and we gives a result of corrector. Our results below should be extended easily to cover time dependent case. The concept of time-harmonic Maxwell's equations can be founded in standard books such as [3, 5].

Two-scale problems are abundant in physics and engineering applications. It is very useful when we study the wave propagation in heterogeneous structures. It is possible to obtain the macroscopic effects caused by processes acting on the microscopic scale, obtained by solving local equations equipped with periodic boundary conditions, which are coupled to macroscopic equations. This procedure is usually known as homogenization named by the fact that we get constant coefficients in the macroscopic effective PEDs, starting with periodic rapidly oscillating coefficients in the original problem. The theoretical foundation of homogenization has developed considerably since the first results in the late 60th by Spagnolo [11]. In the late 80th the two-scale convergence was introduced by Nguetseng [9] and further developed in [1, 2], and many other papers thereafter e.g. in [7, 8] and [10, 11].

In this paper, we will consider a bounded open set simply connected set $\Omega \subset R^3$, which is physical domain. We assume that the boundary $\partial\Omega$ is regular, i.e. $\partial\Omega$ is a once continuously two-dimensional manifold (or $\partial\Omega$ is Lipchitz). The Maxwell's equations read, see [3, 5]:

$$\begin{aligned} &\text{i) } -\frac{\partial D}{\partial t} + curlH = J \\ &\text{ii) } \frac{\partial B}{\partial t} + curlE = 0 \\ &\text{iii) } divD = \rho \\ &\text{iv) } divB = 0 \end{aligned} \qquad (1)$$

where D, E, B, H of $(x, t) \in \Omega \times R$, D is the electric induction, E the electric field, B the magnetic induction and H the magnetic field, $\rho = \rho(x, t)$ the charge density, $J = J(x, t)$ the current density of charges inside, see [5].

The charge and current densities satisfy the charge conservation law $\frac{\partial \rho}{\partial t} + divJ = 0$, as in fact follows from (1).

We assume linear behavior laws:

$$D = \alpha E,\ \ B = \mu H \tag{2}$$

μ, α are the magnetic permeability and electric permittivity respectively and these are assumed constants.

Since we look the solutions in time-harmonic, i.e.

$$D(x, t) = D(x)\ \mathrm{Re}\{\exp \frac{i\omega t}{\varepsilon}\}$$

$$H(x, t) = H(x)\ \mathrm{Re}\{\exp \frac{i\omega t}{\varepsilon}\}$$

$$J(x, t) = J(x)\ \mathrm{Re}\{\exp \frac{i\omega t}{\varepsilon}\} \tag{3}$$

$$B(x, t) = B(x)\ \mathrm{Re}\{\exp \frac{i\omega t}{\varepsilon}\}$$

$$E(x, t) = E(x)\ \mathrm{Re}\{\exp \frac{i\omega t}{\varepsilon}\}$$

The equation (1) becomes in the standard harmonic form see [3, 5]

$$curl\ (curlE\ (x)) + \gamma E\ (x) = F\ (x) \tag{4}$$

where

$$F(x) = -i\omega\mu\, J(x),\ \ \gamma = \mu\omega(-\omega\alpha + i\sigma) \tag{5}$$

To avoid any mathematical problems, we always assume that $\mathrm{Re}(\gamma) > 0, \gamma \in C$.

In this paper, Y-cell will denote the unit cube in R^n, $n \geq 1$ this is the fundamental period of periodic structures, i.e. $Y = (0, l_1)\times\ldots\ldots\times (0, l_n)$, $(l_1, \ldots\ldots, l_n) \in R^n$.

Throughout the paper we consider a sequence $\{\varepsilon_i\}$ of small positive numbers converging to zero which is denoted by $\{\varepsilon\}$. Any subsequence $\{\varepsilon'\}$ of sequences $\{\varepsilon\}$ will also be denoted by $\{\varepsilon\}$. We assume that the material in the domain is ε-periodic in Cartesian coordinate directions (in the sense that it can be viewed as the union of a collection of disjoint open identical cubes with side length ε) (Y-cell). We also introduce the following function spaces see [5]

$$H = L^2\ (\Omega)^3$$

$$H\ (curl, \Omega) = \{E \in H, curlE \in H\} \tag{6}$$

where $curlE = (\partial_{x_2} E_3 - \partial_{x_2} E_{2,} \partial_{x_3} E_1 - \partial_{x_1} E_{3,} \partial_{x_1} E_2 - \partial_{x_2} E_1)$ is the usual rotation of vector fields in R^n, $n \geq 1$. The usual norm of $H\ (curl, \Omega)$ is:

$$\|\ E\ \|_{H\ (curl,\ \ \Omega)} = \|\ E\ \|_{L^2(\Omega)^3} + \|\ curlE\ \|_{L^2(\Omega)^3} \tag{7}$$

Recall that H (*curl, Ω*) is a Hilbert space, see [5]. For tackle to the perfect conductor (type boundary condition), we denote by H_0 (*curl, Ω*) the closed space of H (*curl, Ω*) defined by:

$$V_0 = H_0\ (curl, \Omega) = \{E \in H\ (curl, \Omega),\ \mathrm{n} \wedge \in E(x) = 0 \text{ on } \partial\Omega\} \tag{8}$$

The variational formulation corresponding to problem (4) is given by see [5]:

$\forall\varphi \in H_0(curl, \Omega)$

$a(E,\varphi) = \int_\Omega curlE.curl\,\varphi dx + \gamma \int_\Omega E.\varphi dx = \int_\Omega F.\varphi dx$ (9)

Using Green's formula, the corresponding operator A is characterized by see [3, 5]:

$AE = curl(curlE) + \gamma E$ (10)

Recall that V_0 is a space of distribution and since the sesquilinear form a given above is coercive on $V_0$with the constant $\beta = \inf(1,\gamma)$, there is a unique solution E in V_0 of (4) tanks to Lax-Milgram Lemma see [5].

II. Preliminaries

2.1. *Notations:*

- Let Ω be a bounded open connected set in $R^n, n \geq 1$ and $Y = (0,1)^n$, $n \geq 1$ a unit cube in R^n, which we will call a Y-cell.
- $C^\infty(\Omega)$ is the space of infinitely continuously differentiable functions in Ω and $C_0^\infty(\Omega)$ are the functions in this space with compact support in Ω.
- The symbol # generally represents the periodicity in Y. For example, the space $C_\#^\infty(Y) = \{f \epsilon C^\infty(R^n); f$ is $y-$ *periodic*$\}$.
- Let X be a function space. The $D(\Omega; X)$ denotes the space of $D(\Omega)$ functions with values in X.
- Let $1 \leq P < \infty$ and X be a normed linear space. Then $L^p(\Omega; X)$ denotes the space of measurable functions $f : \Omega \to X$ such that $\int_\Omega ||f||_{||X||}^P < \infty$.
- $H_\#^1(Y)/R$: is defined as the space of equivalence classes with respect to the relation: $u \approx v \Leftrightarrow u - v$ is a constant, $\forall u, v \in H_\#^1(Y)$.
- We say that a function $F : R^n \to R$ is Y-periodic if $F(x + e_i) = F(x)$, $\forall x \in R^n$, and for every $i \in \{1,2,\ldots,n\}$, where $(e_1, \ldots, e_n)$ in the canonical basis of $R^n, n \geq 1$.
- $n \wedge E$, denotes the vector product of $E, n \in R^3$.

2.1. *Theorem [7, 9]:*

A function f belongs to $L^2(\Omega; C_\#(Y))$ if and only if there exists a subset W of measure zero in Ω such that:

a) For any $x \in \Omega \backslash W$, the function $y \to f(x, y)$ is continuous and Y-periodic.

b) For any $y \in Y$, the function $x \to f(x, y)$ is measurable.

c) The function $x \to \sup_y |f(x,y)|$ has finite $L^1(\Omega)$ norm.

2.1. *Remark*

Let f be a function defined on $\Omega \times Y$ such that f is continuous in either of the variables and measurable in the remaining variable. Let f be a Y-periodic function for every fixed $x \in \Omega$. Then it is well-known that f is a caratheody function. As a consequence the function $x \to f(x, \frac{x}{\varepsilon})$ defined on Ω is measurable.

2.2. *Theorem [7, 9]:*

A function $f \in L^2$ $(\Omega;\ L^2_\#$ $(Y;\ R^3))$ is an admissible test function if $f(x,\ \frac{x}{\varepsilon})$ is measurable and $\lim_{\varepsilon\to 0} ||f\left(x, \frac{x}{\varepsilon}\right)||_{L^2(\Omega;\, R^3)} = ||f(x,y)||_{L^2(\Omega\times Y;\, R^3)}$

2.2. *Two Scale Convergence:*

In 1989 Nguetseng [7.9] presented a new concept to homogenize scales of partial differential equations (PDES), the so called two-scale convergence method which was generalized to the $L^p(\Omega)$ -case by Holmbom in [6].

2.2.1. *Definition*

Let $\{E^\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$. On says that $E^\varepsilon \overset{2-s}{\longrightarrow} E_0 = E_0$ $(x,\ y) \in L^2(\Omega\times Y)$ if for any function $\Psi = \Psi(x, y) \in D(\Omega, C^\infty_\#(Y))$, one has:

$$\lim_{\varepsilon\to 0} \int_\Omega E^\varepsilon\, \Psi(x, \frac{x}{\varepsilon}) dx \to \int_\Omega \int_Y E_0\,(x,y)\Psi(x,y) dydx \tag{11}$$

The class of test functions can be enlarged to all admissible test functions defined below [7, 9].

2.2.1. *Theorem*

Let $\{E^\varepsilon\} \in L^2(\Omega)$**.** Suppose that there exists a constant $C > 0$ such that:

$$||E^\varepsilon||_{L^2(\Omega)} \le C \tag{12}$$

then a subsequence (still denoted by ε), can be extracted from $\{E^\varepsilon\}$ such that: $\forall\, \Psi\ \in D(\overline{\Omega},\ C_\#(Y))$

$$\lim_{\varepsilon\to 0} \int_\Omega E^\varepsilon\, \Psi(x, \frac{x}{\varepsilon}) dx \to \int_\Omega \int_Y E_0\,(x,y)\Psi(x,y) dydx \tag{13}$$

where $E_0\ \in L^2(\Omega\times L^2_\#(Y))$. Moreover

$$\lim_{\varepsilon\to 0} \int_\Omega E^\varepsilon v(x)\, \Psi(\frac{x}{\varepsilon}) dx \to \int_\Omega \int_Y E_0\,(x,y) v(x)\Psi(y) dydx \tag{14}$$

for all $v \in C_0(\overline{\Omega},\ )$, $w \in L^2_\#(Y)$.

2.2.1. *Proposition*

Let $\{E^\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, $1 < p \le \infty$. Then up to a subsequence, $\{E^\varepsilon\}$ two-scale converges to $E_0(x,y) \in L^p(\Omega\times Y)$ and converges weakly to $E(x) = \int_Y\ E_0(x,y)\, dy$ in $L^p(\Omega)$. Furthermore E_0 is (uniquely) expressible in the form:

$E_0(x,y) = E(x) + \overline{E_0}\,(x,y)$, with $\int_Y\ \overline{E_0}\,(x,y)dy = 0$. Moreover, if $\overline{E_0}\ \neq 0$ on a subset of $\Omega\times Y$ with positive measure, then the sequence $\{E^\varepsilon\}$ will not converge strongly in $L^p(\Omega)$.

III. Main Result

In this section $R^n = R^3$. We have the following two-scale convergence result:

3.1. *Theorem*

Assume that $F^{\varepsilon} \to F$ weakly in $L^2(\Omega)^3$, and let $\{E^{\varepsilon}\}$ be a solution of problem (4) in $H_0(curl, \Omega)$. Then up to a subsequence, $\{E^{\varepsilon}\}$ two-scale converge in $L^2(\Omega \times Y)^3$ to limit $E_0(x, y)$. Furthermore,

$$curlE^{\varepsilon} \xrightarrow{weakly} curlE(x) \text{ in } L^2(\Omega; R^3)$$

$$curlE^{\varepsilon} \xrightarrow{2-s} curl_x E_0(x,y) + curl_y E_1(x,y) \text{ in } L^2(\Omega; R^3), \text{ with}$$

$$\int_y curl_y E_1 = 0, \ E(x) = \int_Y E_0(x,y)dy \tag{15}$$

where $E = \int_Y E_0(x,y)dy$ is the weak limit in $L^2(\Omega \times R)^3$ of the sequence $\{E^{\varepsilon}\}$ with the boundary condition $n \wedge E(x) = 0$ on $\partial\Omega$ and $E_0(x, y)$, can be posed a

$$E_0(x, y) = E(x) + \nabla_y \Phi(x,y) \tag{16}$$

where Φ is a scalar potential $\Phi : R^3 \to R$

3.1. *Proof of Theorem*

Using the variational formulation (9), we get for all $\Psi \in H_0(curl, \Omega)$ $\int_\Omega curlE^{\varepsilon} curl\Psi dx + \gamma \int_\Omega E^{\varepsilon}\Psi dx = \int_\Omega F^{\varepsilon}\Psi dx$

We get the uniform estimates on E^{ε} and $curlE^{\varepsilon}$, by taking $\Psi = E^{\varepsilon}$, and gets:

$$||E^{\varepsilon}||_V \leq C \tag{17}$$

where C is a constant independent of ε.

Since $\{E^{\varepsilon}\}$ and $\{curlE^{\varepsilon}\}$ are bounded in $L^2(\Omega)^3$, thus up to a subsequence we have: $\forall\Psi \in H_0(curl, \Omega)$

$$E^{\varepsilon} \xrightarrow{weakly} E(x) \text{ in } L^2(\Omega; R^3)$$

$$curlE^{\varepsilon} \xrightarrow{weakly} curlE(x) \text{ in } L^2(\Omega; R^3)$$

Since $\{E^{\varepsilon}\}$ is bounded sequence in $H_0(curl, \Omega)$, we conclude that (up to a subsequence), there exists two vectors $\chi_0(x,y)$, $E_0(x,y)$ in $L^2(\Omega \times R)^3$ such that: $\forall\Psi \in C_0^{\infty}(\Omega, C_{\#}^{\infty}(Y))$;

$$\lim_{\varepsilon\to 0} \int_\Omega E^{\varepsilon}\Psi(x, \frac{x}{\varepsilon})dx \to \int_\Omega \int_Y E_0(x,y)\Psi(x,y)dydx$$

$$\lim_{\varepsilon\to 0} \int_\Omega curlE^{\varepsilon}\Psi(x, \tfrac{x}{\varepsilon})dx \to \int_\Omega \int_Y \chi_0(x,y)\Psi(x,y)dydx \tag{18}$$

It is remains to identify $\chi_0(x.y)$.

Let us chose a test function $\Psi(x.y) \in D(\Omega, C_{\#}^{\infty}(Y))^3$ with $curl_y\Psi = 0$ in, Y, we get:

$$\lim_{\varepsilon\to 0} \int_\Omega curlE^{\varepsilon}\Psi(x,y)dx \to \int_\Omega \int_Y curl_x E_0(x,y)\Psi(x,y)dydx \tag{19}$$

Using stokes theorem and the compact support and (18), we get:

$$(\chi_0(x,y) - curl_x E_0(x,y))\Psi(x,y) = 0 \tag{20}$$

By the decomposition of $(L^2)^3$, see [5], there exists a function $E_1 \in L^2(\Omega, H_\#(curl, Y)$ such that

$$curlE_1 = \chi_0(x,y) - curl_x E_0(x,y) = 0 \tag{21}$$

in $L^2(\Omega, L^2_\#(Y))^3$. Taking the limit (using Stokes theorem and compact support of Ψ in x as well in y in Y) of

$\varepsilon \int_\Omega curlE^\varepsilon \, \Psi(x, \frac{x}{\varepsilon})dx$

Leads to:

$$0 = \int_\Omega \int_Y [curl_y E_0(x,y)])\,\Psi(x,y)dxdy \text{ in } D'(\Omega \times Y) \tag{22}$$

Which implies that $curlE_0(x, y) = 0$ a.e. in $\Omega \times Y$. Thus we conclude that $E_0(x,y)$ is a gradient with respect to the variable y for some scalar valued function $\bar{E}$, i.e. : $E_0(x,y) = \nabla_y \bar{E}(x,y)$. Using proposition (2.2.1), we conclude that $E_0(x,y)$ can be written see [12, 13], as:

$$E_0(x,y) = E(x) + \nabla_y \bar{E}(x,y) \tag{23}$$

for some scalar potential $\bar{E} : R^3 \to R$, and where $E(x) = \int_Y E_0(x,y)dy$. The proof is complete.

3.1. *Remark*

If we change the boundary condition $n \wedge E(x) = 0$ by the non-homogeneous condition $n \wedge E(x) = n \wedge g$, written also as $n \wedge (E - g(x)) = 0$ where g is given smooth vector field, we have the same type of result by setting : $E - g(x) = h(x)$.

We will proof the following new result on correctors:

3.2. *Theorem (Correctors):*

Assume that $F^\varepsilon \to F$ weakly in $L^2(\Omega)^3$, and let $\{E^\varepsilon\}$ be a solution of problem (4) in $H_0(curl, \Omega)$. Let E, E_1 be the solution of the homogenized problem. If $E_0, E_1, curl_x E_0, curl_y E_0, curl_x E_1, curl_y E_1$ are admissible test functions then:

$$\lim_{\varepsilon \to 0} || E^\varepsilon(x) - E_0(x, \tfrac{x}{\varepsilon}) - \varepsilon E_1(x, \tfrac{x}{\varepsilon})||_{H(curl,\Omega)} = 0 \tag{24}$$

where

$$E_0(x,y) = E(x) - \nabla_y \chi_e(y) E(x) \tag{25}$$

and

$$\nabla_y \chi_e(y) = \sum_{i=1}^{3} \chi_e^i(y) e_i \tag{26}$$

$\chi_e^i(y)$ in $H^1_\#(Y)$, solve the local problem of (9).

3.2. *Proof of Theorem*

Using (12) and Theorem (3.1), we conclude that

$E^\varepsilon \to E_0(x,y) + curl_y E_1(x,y), \quad \int_Y curl_y E_1 = 0,\ E(x) = \int_Y E_0(x,y)dy$

$curl E^\varepsilon \to curl_x E_0(x,y) + curl_y E_1(x,y), \quad curl_y E_1(x,y) = 0$ (27)

Using the sesquilinear form (10), and the coercively assumption, one has:

$C \parallel E(x)||^2_{H(curl,\Omega)} \leq \text{Re} A^\varepsilon(E,E)$ (28)

which gives:

$C \|E^\varepsilon(x) - E_0(x,\frac{x}{\varepsilon}) - \varepsilon E_1(x,\frac{x}{\varepsilon})||^2_{H(curl,\Omega)} \leq B_1^\varepsilon + B_2^\varepsilon$ (29)

where

$B_1^\varepsilon := \text{Re}\ A^\varepsilon(E^\varepsilon(x), D_\varepsilon(x))$

$B_2^\varepsilon := \text{Re}\ A^\varepsilon(E_0(x,\frac{x}{\varepsilon}) + \varepsilon E_1(x,\frac{x}{\varepsilon}), D_\varepsilon(x))$ (30)

where for short, we denote: $D^\varepsilon(x) = E^\varepsilon(x) - E_0(x,\frac{x}{\varepsilon}) - \varepsilon E_1(x,\frac{x}{\varepsilon})$. Using the assumptions on $D_\varepsilon(x)$, we get:

$D_\varepsilon(x) \overset{2-s}{\longrightarrow} 0$

$curl D_\varepsilon \overset{2-s}{\longrightarrow} 0$ (31)

Since

i) $E^\varepsilon \overset{2-s}{\longrightarrow} E_0(x,y)$,

ii) $curl E^\varepsilon \overset{2-s}{\longrightarrow} curl_x E_0(x,y) + curl_y E_1(x,y)$, where $curl_y E_1(x,y) = 0$

iii) $n \wedge E_0(x,y) = 0$ on $y \in \partial Y^*$

From the results [12, 13, and 14], we conclude ($\chi_e(y)$ the characteristic function of Y), that:

$E_0(x,y) = E(x) - \nabla_y \chi_e(y)\ E(x)$

$\chi_e(y) = \sum_{i=1}^3 \chi_e^i(y)\, e_i, \quad \chi_e^i \in H^1_\#(Y)$ (32)

solve the local problem of (9). Analyzing the first term B_1 of (29), one has: $B_1^\varepsilon = \text{Re}\ A^\varepsilon(E^\varepsilon(x), D_\varepsilon(x)) = \int_\Omega curl E^\varepsilon . curl D_\varepsilon dx + \gamma \int_\Omega E^\varepsilon . D_\varepsilon dx$

Since, $curl E^\varepsilon \in L^2(\Omega; R^3)$, $curl D_\varepsilon \in L^2(\Omega; R^3)$, passing to limit leads to:

$D_\varepsilon \overset{weakly}{\longrightarrow} 0$ in $L^2(\Omega; R^3)$,

$curl D_\varepsilon \overset{weakly}{\longrightarrow} 0$ in $L^2(\Omega; R^3)$,

Thus $\lim_{\varepsilon\to 0} B_1^\varepsilon = 0$. For the second term of (29) we have:

$B_2^\varepsilon = -\text{Re}\ A^\varepsilon(E_0(x,\frac{x}{\varepsilon}) + \varepsilon E_1(x,\frac{x}{\varepsilon}), D^\varepsilon(x)) =$

$-\int_\Omega curl\{E_0(x,\frac{x}{\varepsilon}) + \varepsilon E_1(x,\frac{x}{\varepsilon})\} curl D_\varepsilon dx - \gamma \int_\Omega \{E_0(x,\frac{x}{\varepsilon}) + \varepsilon E_1(x,\frac{x}{\varepsilon})\}\, D_\varepsilon dx$ (34)

As $curl_x E_0(x,\frac{x}{\varepsilon}) + \varepsilon curl_x E_1, curl_x E_1(x,\frac{x}{\varepsilon}) + curl_y E_1(x,\frac{x}{\varepsilon})$, are admissible test functions, we get:

$$\lim_{\varepsilon\to 0} B_1^{\varepsilon} \xrightarrow{weakly} 0 \text{ in } L^2(\Omega, R^3) \quad (35)$$

which proves the Theorem (3.2).

V. Conclusion

We have studied the Maxwell's equations in time-harmonic form, and we used the two-scale convergence for its. Moreover, we prove a corrector result, which makes the convergence of sequence $\{E^{\varepsilon}\}$ strength to the two-scale convergence E_0.

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