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On the Complete Elliptic Integrals and Babylonian Identity VI: The Complete Elliptic Integral of Second Kind

Edigles Guedes
Number Theorist, Brazil

Prof. Dr. K. Raja Rama Gandhi
Resource perosn in Mathematics for
Oxford University Press and Professor at BITS-Vizag

Abstract **– We use the Babylonian identity to prove some formulaes for the complete elliptic integral of second kind.**

I. Introduction

Using the Babylonian identity, we demonstrated the following formulas, among others:

$$\frac{2E(k)}{\pi}=\sum_{n=0}^{\infty}(-1)^n\binom{1/2}{n}\binom{2n}{n}\left(\frac{k}{2}\right)^{2n}$$

and

$$\frac{E(k)}{\sqrt{\pi}}=\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n}\frac{k^{2n}}{n!^2},$$

for $0<k<1$.

II. Lemmas

Lemma 1. For a and b any number, then

(1) $\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}$

and

(2) $\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}.$

*Proof.*we knew the Babylonian identity [1, page 119]

(3) $ab=\frac{1}{4}[(a+b)^2-(a-b)^2].$

Make the following algebraic manipulation in (3)

$$ab=\left(\frac{a+b}{2}\right)^2\left[1-\left(\frac{a-b}{a+b}\right)^2\right],$$

hence,

$$a^{\frac{1}{2}}b^{\frac{1}{2}}=\frac{a+b}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}\Leftrightarrow\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=\frac{1}{2}\sqrt{1-\left(\frac{a-b}{a+b}\right)^2},$$

and inverting both members, we have

$$a^{-\frac{1}{2}}b^{-\frac{1}{2}}=\frac{2}{a+b}\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}\Leftrightarrow\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\frac{1}{\sqrt{1-\left(\frac{a-b}{a+b}\right)^2}}.$$

Lemma 2. For a and b any number, then

(4) $$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}=-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2(2k-1)}\left(\frac{a-b}{a+b}\right)^{2k}$$

and

(5) $$\frac{a+b}{a^{\frac{1}{2}}b^{\frac{1}{2}}}=2\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}\left(\frac{a-b}{a+b}\right)^{2k}.$$

Proof. we calculate

(6) $$\sqrt{1-z^2}=-\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2(2k-1)}z^{2k}$$

and

(7) $$\frac{1}{\sqrt{1-z^2}}=\sum_{k=0}^{\infty}\frac{(2k)!}{4^k k!^2}z^{2k}$$

Take $z=\frac{a-b}{a+b}$ in (6) and (7), then replace in (1) and (2) respectively, completing the proof.

III. Theorems

Theorem 1. I have

$$E(k)=\frac{1}{2\sqrt{2}}\int_0^{\pi}\sqrt{2-k^2(1-\cos t)}\,dt,$$

where$E(k)$ is the complete elliptic integral of second kind.

Proof. Put $\frac{a-b}{a+b}=t$ in (5), we encounter

(8) $$\frac{1}{\sqrt{1-t^2}}=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}t^{2n}.$$

Multiplying (8) by $\sqrt{1-k^2t^2}$ and integrating from 0 at 1 in t, we find

$$\int_0^1\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_0^1 t^{2n}\sqrt{1-k^2t^2}\,dt\Leftrightarrow$$

(9) $E(k)=\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(2n+1)n!^2}\,{}_2F_1\left(-\frac{1}{2},n+\frac{1}{2};n+\frac{3}{2};k^2\right).$

On the one hand, in [2, page 21], we have

(10) $${}_2F_1(a,b;c;z)=\frac{2^{1-c}\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^{\pi}\frac{(\sin t)^{2b-1}(1+\cos t)^{c-2b}}{\left(1-\frac{1}{2}z+\frac{1}{2}z\cos t\right)^a}dt,$$

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for $\Re(c) > R(b) > 0$. Substituting (10) in (9), we encounter

$$E(k) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} \frac{2^{-n-\frac{1}{2}}\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma(1)} \int_0^{\pi} \frac{(\sin t)^{2n}(1+\cos t)^{-n+\frac{1}{2}}}{\left(1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t\right)^{-\frac{1}{2}}} dt$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi} \sqrt{(1+\cos t)\left(1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t\right)} \sum_{n=0}^{\infty} \frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)(\sin t)^{2n}}{2^{3n}(2n+1)n!^2\,\Gamma\left(n+\frac{1}{2}\right)(1+\cos t)^n} dt$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi} \sqrt{(1+\cos t)\left(1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t\right)} \frac{1}{\sqrt{2}\sqrt{\cos(t)+1}} dt$$

$$= \frac{1}{2} \int_0^{\pi} \sqrt{1-\frac{1}{2}k^2+\frac{1}{2}k^2\cos t}\, dt$$

$$= \frac{1}{2\sqrt{2}} \int_0^{\pi} \sqrt{2-k^2(1-\cos t)}\, dt. \square$$

Theorem 2. we have

$$E(k) = k' E\left(i\frac{k}{k'}\right).$$

Proof. We leave to the reader.

Theorem 3.*For* $0 < k < 1$, *then*

$$\frac{E(k)}{\sqrt{\pi}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \Gamma\left(n+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_n} \frac{k^{2n}}{n!^2},$$

whereE(k) is the complete elliptic integral of second kind.

Proof. We consider

$$E(k) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2}\, {}_2F_1\left(-\frac{1}{2}, n+\frac{1}{2}; n+\frac{3}{2}; k^2\right)$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(2n+1)n!^2} \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r \left(n+\frac{1}{2}\right)_r}{\left(n+\frac{3}{2}\right)_r r!} k^{2r}$$

$$= \sum_{r=0}^{\infty} \left(-\frac{1}{2}\right)_r \left(\sum_{n=0}^{\infty} \frac{(2n)!\left(n+\frac{1}{2}\right)_r}{2^{2n}(2n+1)n!^2\left(n+\frac{3}{2}\right)_r}\right) \frac{k^{2r}}{r!}$$

$$= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_r \Gamma\left(r+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_r \Gamma(r+1)} \frac{k^{2r}}{r!}$$

$$= \sqrt{\pi} \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_r \Gamma\left(r+\frac{3}{2}\right)}{\left(\frac{3}{2}\right)_r} \frac{k^{2r}}{r!^2}.$$

Multiplying both sides by $\frac{1}{\sqrt{\pi}}$ and $r \to n$, then, the result follows.

Theorem 4. For $0 < k < 1$, then

$$\frac{2E(k)}{\pi} = \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \binom{2n}{n} \left(\frac{k}{2}\right)^{2n},$$

where $E(k)$ is the complete elliptic integral of second kind.

Proof. we consider

$$E(k) = \frac{1}{2\sqrt{2}} \int_0^{\pi} \sqrt{2-k^2(1-\cos t)}\, dt$$

$$= \frac{1}{2} \int_0^{\pi} \sqrt{1-\frac{k^2(1-\cos t)}{2}}\, dt$$

$$= \frac{1}{2} \int_0^{\pi} \sum_{n=0}^{\infty} 2^{-n} \binom{1/2}{n} (\cos(t)-1)^n k^{2n}\, dt$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \binom{1/2}{n} \int_0^{\pi} (\cos(t)-1)^n\, dt\, k^{2n}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} 2^{-n}(-1)^n \binom{1/2}{n} \frac{2^n\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} k^{2n}$$

$$= \frac{\sqrt{\pi}}{2} \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \frac{\Gamma\left(n+\frac{1}{2}\right)}{n!} k^{2n}$$

$$= \frac{\pi}{2} \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \frac{(2n)!}{n!^2\, 2^{2n}} k^{2n}$$

$$= \frac{\pi}{2} \sum_{n=0}^{\infty} (-1)^n \binom{1/2}{n} \binom{2n}{n} \left(\frac{k}{2}\right)^{2n}.$$

Multiplying both sides by$\frac{2}{\pi}$, the result follows. □

Theorem 5. For $0 < k < 1$, then

$$E(k) = -\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}(4n^2-1)n!^3} k^{2n},$$

where$E(k)$ is the complete elliptic integral of second kind.

Proof. Put $\frac{a-b}{a+b} = t$ in (5), we encounter

$$(11)\ \frac{1}{\sqrt{1-t^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} t^{2n}.$$

Multiplying (11) by $\sqrt{1-k^2t^2}$ and integrating from 0 at 1 in t,we find

$$\int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!^2} \int_0^1 t^{2n}\sqrt{1-k^2t^2}\, dt.$$

Let $a = 1+kt$ and $b = 1-kt$ in (4)

$$(13)\sqrt{1-k^2t^2} = -\sum_{m=0}^{\infty} \frac{(2m)!}{4^m m!^2(2m-1)} k^{2m}t^{2m}$$

We put (13) into (12)

$$E(k) = -\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\int_0^1 t^{2n}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2\,(2m-1)}k^{2m}t^{2m}\,dt$$

$$= -\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2\,(2m-1)}\left[\int_0^1 t^{2(m+n)}\,dt\right]k^{2m}$$

$$= -\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!^2}\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2\,(2m-1)}\frac{k^{2m}}{2m+2n+1}$$

$$= -\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2\,(2m-1)}\left[\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(2m+2n+1)n!^2}\right]k^{2m}$$

$$= -\sum_{m=0}^{\infty}\frac{(2m)!}{2^{2m}m!^2\,(2m-1)}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{2m+3}{2}\right)}{(2m+1)\Gamma(m+1)}\right]k^{2m}$$

$$= -\sqrt{\pi}\sum_{m=0}^{\infty}\frac{(2m)!\,\Gamma\left(m+\frac{3}{2}\right)}{2^{2m}(4m^2-1)m!^3}k^{2m}.$$

Let $m \to n$ in (14);this concludes the proof.□

Theorem 6. For $0 < k < 1$, then

$$E(k) + \frac{\pi}{32}k^2\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;k^2\right) + \frac{3\pi}{32}k^2\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;k^2\right) = \frac{\pi}{2},$$

where $E(k)$ is the complete elliptic integral of second kind and ${}_\alpha F_\beta(a,b,c,d;e,f,g;z)$ is the generalized hypergeometric series.

Proof. Using the identity

$$(11)\ \frac{1}{4n^2-1} = \frac{1}{4n(2n+1)} + \frac{1}{4n(2n-1)}.$$

into the Theorem 5, we encounter

$$E(k) = \frac{\pi}{2} - \frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}n(2n+1)n!^3}k^{2n} - \frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}n(2n-1)n!^3}k^{2n}.$$

we calculate

$$\frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}n(2n+1)n!^3}k^{2n} = \frac{\pi}{32}k^2\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;k^2\right)$$

and

$$\frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}\frac{(2n)!\,\Gamma\left(n+\frac{3}{2}\right)}{2^{2n}n(2n-1)n!^3}k^{2n} = \frac{3\pi}{32}k^2\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;k^2\right).$$

From (12), (13) and (14), I obtain

$$E(k) = \frac{\pi}{2} - \frac{\pi}{32}k^2\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;k^2\right) - \frac{3\pi}{32}k^2\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;k^2\right),$$

ergo,

$$E(k) + \frac{\pi}{32}k^2\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;k^2\right) + \frac{3\pi}{32}k^2\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;k^2\right) = \frac{\pi}{2}.\ \square$$

Note: For the reader's delight, we construct the Table 1.

Corollary 1. For $0 < k < 1$, then

$$\frac{\pi}{32}k^2\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;k^2\right) + \frac{3\pi}{32}k^2\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;k^2\right)$$
$$= K(k)\left[E\left(\sqrt{1-k^2}\right) - K\left(\sqrt{1-k^2}\right)\right] - E(k)\left[1 - K\left(\sqrt{1-k^2}\right)\right],$$

where $K(k)$ denotes the complete elliptic integral of first kind and $E(k)$denotes the complete elliptic integral of second kind and ${}_\alpha F_\beta(a,b,c,d;e,f,g;z)$ is the generalized hypergeometric series.

Proof. In [3], we knew the Legendre's relation

$$K(k)E\left(\sqrt{1-k^2}\right) + E(k)K\left(\sqrt{1-k^2}\right) - K(k)K\left(\sqrt{1-k^2}\right) = \frac{\pi}{2}.$$

From Theorem 6 and (15), we get the desired result.

References

[1] Havil, Julian, *Gamma: Exploring the Euler's Constant*, Princeton University Press, 2003.

[2] Slater, Lucy Joan, *Generalized Hypergeometric Functions*, Cambridge University Press, 1966.

[3] http://en.wikipedia.org/wiki/Elliptic_integral, available in July 02, 2012.

[4] Armitage, J. V. and Eberlein, W. F., *Elliptic Functions*, Cambridge University Press, 2006.

Table 1: In this table, we have: first column: m; second column: $k = 1/m$; third column: $E\left(\frac{1}{m}\right)$; fourth column: $\frac{\pi}{32m^2}\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;\frac{1}{m^2}\right)$, fifth column: $\frac{3\pi}{32m^2}\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;\frac{1}{m^2}\right)$; sixth column: $E\left(\frac{1}{m}\right)+\frac{\pi}{32m^2}\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;\frac{1}{m^2}\right)+\frac{3\pi}{32m^2}\,{}_4F_3\left(\frac{1}{2},1,1,\frac{5}{2};2,2,2;\frac{1}{m^2}\right)$ and seventh column: $\pi/2$.

2	$\frac{1}{2}$	1.46746220933942	0.0265034758034005	0.076830641652069	1.57079632679489	1.57079632679489
3	$\frac{1}{3}$	1.52620923421218	0.0112679338274853	0.0333191587552238	1.57079632679489	1.57079632679489
4	$\frac{1}{4}$	1.54595725610546	0.00624701881660543	0.0185920518728261	1.57079632679489	1.57079632679489
5	$\frac{1}{5}$	1.55496854624253	0.00397200687915803	0.0118557736732093	1.57079632679489	1.57079632679489
6	$\frac{1}{6}$	1.55983053712868	0.00274866068320246	0.00821712898301137	1.57079632679489	1.57079632679489
7	$\frac{1}{7}$	1.56275112926275	0.00201517675314221	0.00603002077899999	1.57079632679489	1.57079632679489
8	$\frac{1}{8}$	1.56464230926255	0.00154077112032801	0.00461324641201171	1.57079632679489	1.57079632679489
9	$\frac{1}{9}$	1.56593690932296	0.0012162668805588	0.00364315059137878	1.57079632679489	1.57079632679489
10	$\frac{1}{10}$	1.56686194202166	0.000984521726720624	0.0029498630465077	1.57079632679489	1.57079632679489
11	$\frac{1}{11}$	1.56754583334049	0.000813254902375769	0.00243723855202368	1.57079632679489	1.57079632679489
12	$\frac{1}{12}$	1.56806568864347	0.000683105117812652	0.00204753303361477	1.57079632679489	1.57079632679489
13	$\frac{1}{13}$	1.56847007918414	0.000581885219496044	0.00174436239125746	1.57079632679489	1.57079632679489
14	$\frac{1}{14}$	1.56879083928819	0.000501612142581216	0.00150387536412359	1.57079632679489	1.57079632679489
15	$\frac{1}{15}$	1.56904954040188	0.000436878853516082	0.00130990753949578	1.57079632679489	1.57079632679489
16	$\frac{1}{16}$	1.56926122065336	0.000383917281028132	0.00115118886050603	1.57079632679489	1.57079632679489
17	$\frac{1}{17}$	1.56943662358195	0.000340036213975309	0.00101966699896408	1.57079632679489	1.57079632679489
18	$\frac{1}{18}$	1.56958359027989	0.000303271955457257	0.00090946455954155	1.57079632679489	1.57079632679489
19	$\frac{1}{19}$	1.56970795205341	0.000272164418968319	0.000816210322517023	1.57079632679489	1.57079632679489
20	$\frac{1}{20}$	1.56981411841638	0.000245609698924618	0.000736598679583975	1.57079632679489	1.57079632679489
21	$\frac{1}{21}$	1.56990547372564	0.000222760652382764	0.000668092416871503	1.57079632679489	1.57079632679489
22	$\frac{1}{22}$	1.56998465046609	0.000202958417212874	0.000608717911587038	1.57079632679489	1.57079632679489
23	$\frac{1}{23}$	1.57005372116044	0.000185684332888638	0.000556921301567939	1.57079632679489	1.57079632679489
24	$\frac{1}{24}$	1.57011433546505	0.00017052560050744	0.00051146572933598	1.57079632679489	1.57079632679489
25	$\frac{1}{25}$	1.57016781964284	0.000157150370922228	0.000471356781128718	1.57079632679489	1.57079632679489
26	$\frac{1}{26}$	1.57021524976737	0.000145289414296077	0.000435787613221675	1.57079632679489	1.57079632679489
27	$\frac{1}{27}$	1.57025750629788	0.000134722456195167	0.000404098040816368	1.57079632679489	1.57079632679489
28	$\frac{1}{28}$	1.57029531525301	0.000125267870132483	0.000375743671752895	1.57079632679489	1.57079632679489
29	$\frac{1}{29}$	1.57032927961663	0.000116774816221124	0.000350272362043396	1.57079632679489	1.57079632679489
30	$\frac{1}{30}$	1.57035990353722	0.000109117184258011	0.00032730607341589	1.57079632679489	1.57079632679489
31	$\frac{1}{31}$	1.5703876111506	0.000102188882918962	0.000306526761374074	1.57079632679489	1.57079632679489
32	$\frac{1}{32}$	1.57041276134927	0.0000959001436802174	0.000287665301946885	1.57079632679489	1.57079632679489