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Stability and Bifurcation Analysis of a Mechanical Vibration System with Clearance

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Abstract **– The model of a three-dimension mechanical collision system with clearances was established in this paper. By using Matrix modal analysis and Poincaré map, stability and bifurcation of this system are investigated. Numerical simulations are also given, which confirm the analytical results.**

Keywords **– Mechanical Collision Systems, Bifurcation, Poincare Mapping.**

I. Introduction

In engineering practice, multi-degree-of-freedom mechanism with clearance is ubiquitous. The existence of clearance will cause friction between components of mechanical system during operation, lead to equipment wear failure, so that it will reduce the operation efficiency of equipment, and even produce huge potential safety hazards. For example, the reliability and durability of the system are often determined by the clearance in parts assembly; the clearance between gears, connecting rods, bearings and other transmission parts will have a great impact on the transmission efficiency. However, some mechanical equipment are based on the principle of collision, such as vibrating sand blasting machine, impact vibration shaping machine, vibrating screen and so on. The dynamic properties of the motion of these devices are often very complex. The reason for the complexity is the gap between the parts, which leads to collision. In addition, some non-smooth factors make the bifurcation and chaos phenomena of the motion system more changeable. Therefore, the research of vibration system with clearance has a broad research prospect.

At present, the research on dynamic characteristics of vibro-impact system with clearance is a hot issue. The vibro-impact between oscillators will lead to strong discontinuity and non-linearity of the whole mechanical system, thus its dynamic characteristics will appear complex and changeable. Li and Tian[1] established a kind of single-degree-of-freedom vibro-impact system based on vehicle leaf spring system. The Poincare mapping method was used to analyze the system. It was found that Hopf bifurcation, almost periodic bifurcation and doubling periodic bifurcation may occur in the process of the system motion. Zhao et al. [2] analyzed the collision model of bus axles, and extracted a kind of two-degree-of-freedom bilateral collision model. It was presented there were non-degenerate Hopf bifurcations and period doubling bifurcations in this kind of system, which gradually moved to chaos. Shaw[3] took friction into account in the dynamic model and studied a class of piece-wise linear collision system with single degree of freedom and the stability of its periodic motion. It was illustrated that under different friction conditions, the system would appear pichfork bifurcation, Hopf bifurcation and periodic doubling bifurcation. Luo et al.[4] considered the relationship between the bifurcation parameters and the dynamic behavior of the system, and study the periodic motion of a two-degree-of-freedom vibro-impact system. The force, gap size and frequency were taken as bifurcation parameters, respectively. Hopf bifurcation, pichfork bifurcation, redundant bifurcation and almost periodic bifurcation were obtained, which indicated the diversity of periodic collision systems. Luo et al.[5] summarized the research methods of this kind of problem, mainly including the paradigm theory, modal analysis and Poincaré mapping method.

In this paper, a three-degree-of-freedom mechanical vibro-impact model with clearance was established. Using the matrix modal analysis method, the approximate analytical solution of the model is obtained. Then, considering the perturbed periodic motion of the model, the expression of periodic motion is derived from the known momentum transformation equation and boundary conditions. According to the expression, Poincare mapping and corresponding linearization matrix are obtained. The type of bifurcation is judged by the eigenvalues of the linearization matrix. Finally, numerical simulations are given to verify the analytic results.

II. Mechanical Model and Equation of Motion

Consider a three-degree-of-freedom collision model with clearance as shown in Figure 1.

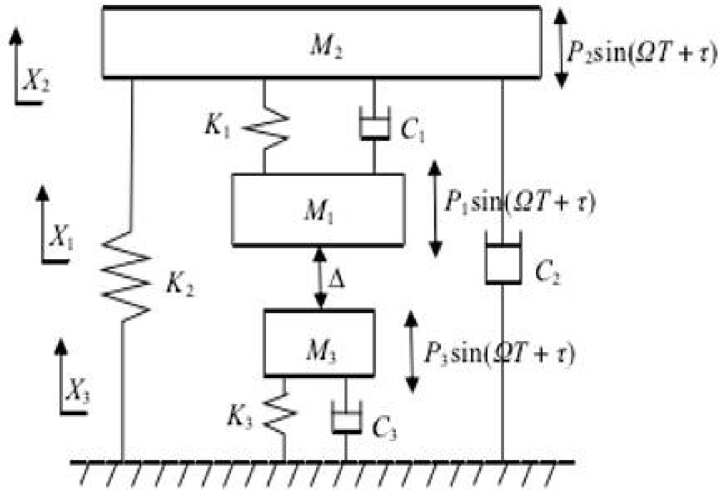

Fig. 1. Three-degree-of-freedom vibro-imapct model with clearance.

The masses of the three oscillators are M_1, M_2, M_3, respectively, which are connected by the linear spring with stiffness K_1, K_2, K_3 and linear dampers with resistance coefficients C_1, C_2, C_3. Each oscillator moves in the y direction and is subject to periodic forces $P_i \sin(\Omega T + \tau)$, $\mathrm{i} = 1, 2, 3$.

Observing the motion of the oscillator, one can see that the collision occurs when the displacement difference between the oscillator M_1 and the oscillator M_3 is the gap Δ. After collision, the velocities of the three oscillators change, and then they move at new initial velocities, colliding again, forming periodic motion.

Assume that if the damping in the model is proportional, energy loss will occur during the collision process. These losses are determined by the collision recovery coefficient R, and the collision duration is neglected. Before the oscillator collides, the dimensionless differential equation of the motion of M_1, M_2 can be expressed as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & \mu_{m_2} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + 2\xi \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1+\mu_{k_2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_{10} \\ f_{20} \end{Bmatrix} \sin(\omega t + \tau) \tag{1}$$

The differential equation of the motion of M_3 can be expressed as:

$$\mu_{m_3}\ddot{x}_3 + 2\xi\mu_{c3}\dot{x}_3 + \mu_{k3}x_3 = f_{30}\sin(\omega t + \tau) \tag{2}$$

When $x_1 - x_3 = \delta$, the impact equation of block M_1, M_2 is as follows:

$$\dot{x}_{1-} + \mu_{m3}\dot{x}_{3-} = \dot{x}_{1+} + \mu_{m3}\dot{x}_{3+} \tag{3}$$

Collision recovery coefficient R satisfies

$$\mathrm{R}=\frac{\dot{x}_{3+}-\dot{x}_{1+}}{\dot{x}_{1-}-\dot{x}_{3-}} \tag{4}$$

Where $\dot{x}_{1-},\dot{x}_{1+},\dot{x}_{3-},\dot{x}_{3+}$ denote the impact velocity in the forward and afterward collision time.

Dimensionless constants are as follows:

$$\mu_{mi}=\frac{M_i}{M_1}、\mu_{ci}=\frac{C_i}{C_1}、\mu_{ki}=\frac{K_i}{K_1}、\xi=\frac{C_1}{2\sqrt{K_1M_1}}、x_i=\frac{X_iK_1}{P_0}、f_{i0}=\frac{P_i}{P_0}、\omega=\Omega\sqrt{\frac{M_1}{K_1}}\quad t=\mathrm{T}\sqrt{\frac{M_1}{K_1}}$$

$$\delta=\frac{\Delta\cdot K_1}{P_0}、P_0=\sqrt{P_1^2+P_2^2+P_3^2},\qquad i=1,2,3$$

Where $\dot{x}_i$ and $\ddot{x}_i$ denote the first derivative and the second derivative of the displacement x_i of the oscillator M_i for time t are expressed respectively.

Let $\omega_{n1},\omega_{n2},\omega_{n3}$ denote the frequency of the system under undamped free vibration. Because the motion of the block M_3 is a single-degree-of-freedom vibro-impact system, $\omega_{n3}=\sqrt{\frac{\mu_{k_3}}{\mu_{m_3}}}$.

Thus, one can get the general solution of equation[6] (2):

$$x_3=e^{-\eta_3 t}(a_3cos\omega_{d3}t+b_3sin\omega_{d3}t)+A_3\sin(\omega t+\tau)+B_3\cos(\omega t+\tau) \quad (5a)$$

$$\dot{x}_3=e^{-\eta_3 t}[(b_3\omega_{d3}-\eta_3a_3)cos\omega_{d3}t-(a_3\omega_{d3}+\eta_3b_3)sin\omega_{d3}t]+A_3\,\omega\cos(\omega t+\tau)-B_3\omega\sin(\omega t+\tau) \quad (5b)$$

Where $\eta_3=\zeta\omega_{n3}^2,\omega_{d3}=\sqrt{\omega_{n3}^2-\eta_3^2}$.

Block M_1 and M_2 constitute a two-degree-of-freedom vibro-impact system. The general solution of equation (1) is obtained by modal analysis method.

Let Ψ denote the regular modal matrix of equation (1). Take Ψ as the transformation matrix and do the following coordinate transformation: $\mathrm{X}=\Psi\xi$, where $\mathrm{X}=(x_1,x_2)^T;\xi=(\xi_1,\xi_2)^T$.

After coordinate transformation, equation (1) can be decoupled into

$$\mathrm{I}\ddot{\xi}+C\dot{\xi}+\Lambda\xi=\bar{F}\sin(\omega t+\tau)$$

Where I is a 2 x 2 order unit matrix , C and_Λ are 2 x 2 order diagonal matrices.

$$\mathrm{C}=2\zeta\Lambda=\mathrm{diag}[2\zeta\omega_{n1}^2,2\zeta\omega_{n2}^2],\Lambda=\mathrm{diag}[\omega_{n1}^2,\omega_{n2}^2]$$

$$\bar{F}=(\bar{f_1},\bar{f_2})^T=\Psi^TP,P=(1-f_{20},f_{20})^T$$

The general solution of equation (1) can be obtained by modal superposition method.

$$x_i=\sum_{j=1}^{2}\psi_{ij}(e^{-\eta_j t}(\tilde{a}_jcos\omega_{dj}t+\tilde{b}_jsin\omega_{dj}t)+A_j\sin(\omega t+\tau)+B_j\cos(\omega t+\tau)) \quad (6a)$$

$$\dot{x}_i=\sum_{j=1}^{2}\psi_{ij}\left(e^{-\eta_j t}\left[(\tilde{b}_j\omega_{dj}-\eta_j\tilde{a}_j)cos\omega_{dj}t-(\tilde{a}_j\omega_{dj}+\eta_j\tilde{b}_j)sin\omega_{dj}t\right]+A_j\,\omega\cos(\omega t+\tau)-B_j\omega\sin(\omega t+\tau)\right) \quad (6b)$$

Where, ψ_{ij} is the element of the regular modal matrix Ψ,A_j、$B_j,j=1,2,3$ are amplitude coefficient. Substituting (3), (4), (5a), (5b) into (1), (2), one can get the expression of the amplitude coefficient:

$$A_j = \frac{(\omega_{nj}^2 - \omega^2)}{(\omega_{nj}^2 - \omega^2)^2 + 4\zeta^2\omega_{nj}^2\omega^2} F_j, B_j = \frac{-2\zeta\omega_{nj}\omega}{(\omega_{nj}^2 - \omega^2)^2 + 4\zeta^2\omega_{nj}^2\omega^2} F_j \ (j = 1,2,3)$$

III. Poincare Mapping

With appropriate system parameters, periodic collisions occur in the collision system as shown in Figure 1. We use the symbol q = p/n to represent the type of periodic motion, where n denotes the number of force cycles and p denotes the number of collisions, q = 1/n means a collision occurs in n periods. That is, when q = 0, collision occurs, then the next collision occurs when q = 2nπ/ω (n = 1, 2, 3...).

We can obtain the corresponding Poincare mapping by the perturbed motion of the system and the boundary collision conditions.

Define section σ as$\sigma = \{(x_1, \dot{x}_1, x_2, \dot{x}_2, x_3, \dot{x}_3, \theta) \in R^6 \times S, x_1 - x_3 = \delta, \dot{x}_1 = \dot{x}_{1+}, \dot{x}_3 = \dot{x}_{3+}\}, \theta = \omega t \bmod 2\pi$. Taking σ as Poincare cross section, the mapping is established as follows:

$$X' = \tilde{f}(\upsilon, X)$$

Where, υ as bifurcation parameter, $X^* = (x_{10}, \dot{x}_{1+}, x_{20}, \dot{x}_{20}, \dot{x}_{3+}, \tau_0)^T$ represents the stable fixed point on the Poincaré section, , X and X′ satisfy $X = X^* + \Delta X$, $X' = X^* + \Delta X'$. ΔX and ΔX′ denote perturbation in motion. From the previous analysis we can derived, when $\tilde{x}_1 - \tilde{x}_3 \le \delta$, the motion with perturbation of the system can be expressed as follows:

$$x_i = \sum_{j=1}^{2} \psi_{ij}(e^{-\eta_j t}(a_j cos\omega_{dj}t + b_j sin\omega_{dj}t) + A_j\, sin(\omega t + \tau_0 + \Delta\tau) + B_j cos\, \omega t + \tau_0 + \Delta\tau))\ (7a)$$

$$\dot{x}_i = \sum_{j=1}^{2} \psi_{ij}(e^{-\eta_j t}\left[(b_j\omega_{dj} - \eta_j a_j)\, cos\omega_{dj}\, t - (a_j\omega_{dj} + \eta_j b_j)\, sin\omega_{dj} t\right] + A_j\, \omega cos\, (\omega t + \tau_0 + \Delta\tau)$$
$$- B_j\omega\, sin\ (\omega t + \tau_0 + \Delta\tau))\ (7b)$$

For perturbed motion, we assume that when M_1 and M_3 collide, t = 0, when they re-collide, $t_e = \frac{2n\pi + \Delta\theta}{\omega}, \Delta\theta = \Delta\tau' - \Delta\tau$. We can get the boundary conditions of collision motion:

$$\begin{cases} \tilde{x}_1(0) = x_{10} + \Delta x_{10}, & \dot{\tilde{x}}_1(0) = \dot{x}_{10} + \Delta\dot{x}_{10}, & \tilde{x}_1(t_e) = x_{10} + \Delta x'_{10} \\ \dot{\tilde{x}}_1(t_e) = x_{10} + \Delta\dot{x}'_{10}, & \tilde{x}_2(0) = x_{20} + \Delta x_{20}, & \dot{\tilde{x}}_2(0) = \dot{x}_{20} + \Delta\dot{x}_{20} \\ \tilde{x}_2(t_e) = x_{20} + \Delta x'_{20}, & \dot{\tilde{x}}_2(t_e) = x_{20} + \Delta\dot{x}'_{20}, & \tilde{x}_3(0) = x_{30} + \Delta x_{30} \\ \dot{\tilde{x}}_3(0) = \dot{x}_{30} + \Delta\dot{x}_{30}, & \tilde{x}_3(t_e) = x_{30} + \Delta x'_{30}, & \dot{\tilde{x}}_3(t_e) = x_{30} + \Delta\dot{x}'_{30} \\ \tilde{x}_1(0) - \tilde{x}_3(0) = \tilde{x}_1(t_e) - \tilde{x}_3(t_e) = \delta \\ \dot{\tilde{x}}_1(t_{e+}) + \mu_{m_3}\dot{\tilde{x}}_3(t_{e+}) = \mu_{m_3}\dot{\tilde{x}}_3(t_e) + \dot{\tilde{x}}_1(t_e) \end{cases}$$

Substituting the condition t = 0 into (7a), (7b), the parameters to be determined can be obtained.

$$\tilde{a}_1 = \frac{1}{D}(\psi_{22}(x_{10} + \Delta x_{10}) - \psi_{12}(x_{20} + \Delta x_{20}) - DA_1\, sin(\tau_0 + \Delta\tau) - DB_1 cos(\tau_0 + \Delta\tau))$$

$$\tilde{a}_2 = \frac{1}{D}(-\psi_{21}(x_{10} + \Delta x_{10}) + \psi_{11}(x_{20} + \Delta x_{20}) - DA_2\, sin(\tau_0 + \Delta\tau) - DB_2 cos(\tau_0 + \Delta\tau))$$

$$\tilde{a}_3 = x_{10} + \Delta x_{10} - \delta - A_3\, sin(\tau_0 + \Delta\tau) - B_3\, cos(\tau_0 + \Delta\tau)$$

$$\tilde{b}_1 = \frac{1}{D\omega_{d1}}(\psi_{22}((\dot{x}_{1+} + \Delta\dot{x}_{1+}) + \eta_1(x_{10} + \Delta x_{10})) - \psi_{12}(\dot{x}_{20} + \Delta\dot{x}_{20}) + \eta_1(x_{20} + \Delta x_{20}) - D(A_1\omega + \eta_1 B_1)\cos(\tau_0 + \Delta\tau) + D(B_1\omega - \eta_1 A_1)\sin(\tau_0 + \Delta\tau))$$

$$\tilde{b}_2 = \frac{1}{D\omega_{d2}}(-\psi_{21}((\dot{x}_{1+} + \Delta\dot{x}_{1+}) + \eta_2(x_{10} + \Delta x_{10})) + \psi_{11}(\dot{x}_{20} + \Delta\dot{x}_{20}) + \eta_2(x_{20} + \Delta x_{20}) - D(A_2\omega + \eta_2 B_2)\cos(\tau_0 + \Delta\tau) + D(B_2\omega - \eta_2 A_2)\sin(\tau_0 + \Delta\tau))$$

$$\tilde{b}_3 = \frac{1}{\omega_{d3}}((\dot{x}_{3+} + \Delta\dot{x}_{3+}) + \eta_3(x_{10} + \Delta x_{10} - \delta) - (A_3\omega + \eta_3 B_3)\cos(\tau_0 + \Delta\tau) + (B_3\omega - \eta_3 A_3)\sin(\tau_0 + \Delta\tau))$$

$D = \psi_{22}\psi_{11} - \psi_{21}\psi_{12}$ as modules of regular modal matrix

Substituting the condition $t=t_e$ into (7a), (7b), we can get the result as follows:

$$\sum_{j=1}^{2} \psi_{1j}\tilde{\xi}_j(t_e) - \tilde{\xi}_3(t_e) = \delta$$

$$\begin{cases} x_{10} + \Delta x'_{10} = \sum_{j=1}^{2} \psi_{1j}\tilde{\xi}_j(t_e) \\ \dot{x}_{1+} + \Delta\dot{x}'_{1+} = \frac{1}{1+\mu_{m_3}}(\mu_{m_3}(1+R)\dot{\xi}_3(t_e) + (1-R)\mu_{m_3}\sum_{j=1}^{2} \psi_{1j}\dot{\tilde{\xi}}_j(t_e)) \\ x_{20} + \Delta x'_{20} = \sum_{j=1}^{2} \psi_{2j}\tilde{\xi}_j(t_e) \\ \dot{x}_{20} + \Delta\dot{x}'_{20} = \sum_{j=1}^{2} \psi_{2j}\dot{\tilde{\xi}}_j(t_e) \\ \dot{x}_{3+} + \Delta\dot{x}'_{3+} = \frac{1}{1+\mu_{m_3}}((\mu_{m_3} - R)\dot{\xi}_3(t_e) + (1+R)\mu_{m_3}\sum_{j=1}^{2} \psi_{1j}\dot{\tilde{\xi}}_j(t_e)) \\ \Delta\tau' = \Delta\tau + \Delta\theta(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \end{cases}$$

Where

$$\tilde{\xi}_j(t) = e^{-\eta_j t}(a_j\cos\omega_{dj}t + b_j\sin\omega_{dj}t) + A_j\sin(\omega t + \tau_0 + \Delta\tau) + B_j\cos\omega t + \tau_0 + \Delta\tau)$$

$$\dot{\tilde{\xi}}_j(t_e) = e^{-\eta_j t}[(b_j\omega_{dj} - \eta_j a_j)\cos\omega_{dj}t - (a_j\omega_{dj} + \eta_j b_j)\sin\omega_{dj}t] + A_j\,\omega\cos(\omega t + \tau_0 + \Delta\tau) - B_j\omega\sin(\omega t + \tau_0 + \Delta\tau)$$

Let $g(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta) = \sum_{j=1}^{2} \psi_{1j}\tilde{\xi}_j(t_e) - \tilde{\xi}_3(t_e) - \delta$, we can get the conditions for the existence of fixed points:

$$g(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta)|_{(0,0,0,0,0,0,0)} = 0$$

Assuming $(\partial g/\partial\Delta\theta)|_{(0,0,0,0,0,0,0)} \neq 0$, according to the existence theorem of implicit functions, we can obtain:

$$\Delta\theta = \Delta\theta(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta)$$

It satisfy $\Delta\theta(0,0,0,0,0,0) = 0$

Thus, one can get the expression of Poincare mapping [7]:

$$\begin{cases} \Delta x'_{10} = \tilde{f}_1(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta) - x_{10} = h_1(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \\ \Delta x'_{1+} = \tilde{f}_2(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta) - \dot{x}_{1+} = h_2(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \\ \Delta x'_{20} = \tilde{f}_3(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta) - x_{20} = h_3(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \\ \Delta\dot{x}'_{20} = \tilde{f}_4(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta) - \dot{x}_{20} = h_4(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \\ \Delta x'_{3+} = \tilde{f}_5(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau, \Delta\theta) - \dot{x}_{3+} = h_5(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \\ \Delta\tau' = \Delta\tau + \Delta\theta(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) = h_6(\Delta x_{10}, \Delta\dot{x}_{1+}, \Delta x_{20}, \Delta\dot{x}_{20}, \Delta\dot{x}_{3+}, \Delta\tau) \end{cases}$$

Simplify this mapping to $\Delta X' = H(\nu, \Delta X)$

Therefore, the Jacobi matrix mapping at the fixed point can be abbreviated as

$$Df(\nu, 0) = \frac{\partial f(\nu, \Delta X)}{\partial \Delta X}\Big|_{(\nu,0,0,0,0,0,0,)} \tag{8}$$

According to the eigenvalue of matrix (8), we can judge whether the periodic motion of the system is stable or whether there is bifurcation. The type of bifurcation is determined by the eigenvalue crossing circle.

IV. Bifurcation Parameter Analysis

Based on the Poincaré mapping matrix, we calculate the different types of bifurcations that may occur under certain critical values. Let ν_c be the critical value and $X^* = (x_{10}, \dot{x}_{10}, \dot{x}_{2+}, x_{30}, \dot{x}_{30}, \tau_0)^T$ be the 1-1-1 fixed point of the system. Now we assume the eigenvalues of the Poincaré matrix at the critical point as follows.

(i) There is a real eigenvalue 1 and a pair of complex conjugate eigenvalues with modules equal to 1 in the eigenvalue of $Df(\nu, X^*)$.

(ii) The modules of the remaining eigenvalues of $Df(\nu, X^*)$ are less than 1.

To simplify the problem, we take the following coordinate transformation:

$$\omega_1 = v_1 - v_{1c}, \omega_2 = v_2 - v_{2c}, \omega = (\omega_1, \omega_2)^T, X = X^* + P\tilde{Y}$$

Turn the mapping into

$$Y^* = F(\omega; Y) \ (9)$$

Translate Poincare mapping matrix into

$$DF(\omega; 0) = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & Re\lambda_2 & -Im\lambda_2 & 0 \\ 0 & Im\lambda_2 & Re\lambda_2 & 0 \\ 0 & 0 & 0 & D_1 \end{bmatrix} \tag{10}$$

Where, , $\lambda_i = \lambda_i(v_c + \omega), i = 1{,}2$ and D_1 is a block diagonal matrix with eigenvalues $\tilde{\lambda}_4(\omega), \tilde{\lambda}_5(\omega), \tilde{\lambda}_6(\omega)$.

By using the central manifold-norm method, the mapping can be transformed into the real form of the norm $\Phi(Y; \varepsilon)$, and the form with neglecting the higher order terms is as follows:

$$y_1' = y_1 + \varepsilon_1 y_1 + a y_1^2 + \mathrm{b}(y_2^2 + y_3^2) + g y_1^3 + h y_1(y_2^2 + y_3^2) \text{(11a)}$$

$$y_2' = (\alpha + \varepsilon_2)y_2 - (\beta + \varepsilon_3)y_3 + c y_1 y_2 - d y_1 y_3 + e y_1^2 y_2 - f y_1^2 y_3 + m y_2(y_2^2 + y_3^2) - n y_3(y_2^2 + y_3^2) \text{(11b)}$$

$$y_3' = (\beta + \varepsilon_3)y_2 + (\alpha + \varepsilon_2)y_3 + c y_1 y_3 + d y_1 y_2 + e y_1^2 y_3 + f y_1^2 y_2 + m y_3(y_2^2 + y_3^2) + n y_2(y_2^2 + y_3^2) \text{(11c)}$$

where, $\varepsilon_i = \varepsilon_i(\omega), \varepsilon_i(0) = 0$

Since pitchfork bifurcation, saddle-node bifurcation or trans-critical bifurcation may occur near the equilibrium point, in order to discuss the types of bifurcation in different regions, we need to discuss the behavior near each fixed point.

Firstly, the fixed points of the mapping (11) :

$$Y_0^* = (0,0,0)^T, Y_1^* = \left(\frac{-a+\sqrt{a^2-4g\varepsilon_1}}{2g}, 0,0\right)^T, Y_2^* = \left(\frac{-a-\sqrt{a^2-4g\varepsilon_1}}{2g}, 0,0\right)^T$$

If $\mathrm{a} = 0, \mathrm{b} = 0, \frac{\varepsilon_1}{g} < 0$, three fixed points become:

$$Y_0^* = (0,0,0)^T, Y_1^* = \left(\sqrt{-\frac{\varepsilon_1}{g}}, 0,0\right)^T, Y_2^* = \left(-\sqrt{-\frac{\varepsilon_1}{g}}, 0,0\right)^T$$

It is easy to get that Y_0^* is a trivial fixed point, while Y_1^* and Y_2^* are periodic 1 fixed points generated by forked bifurcation. And they are symmetrical about the origin, so we only need to discuss the behavior of one of them. The linearization matrices of mapping_$\Phi(\mathrm{Y};\varepsilon)$ at fixed points Y_0^* and Y_1^* are respectively:

$$B_0 = \begin{bmatrix} 1+\varepsilon_1 & 0 & 0 \\ 0 & \alpha+\varepsilon_2 & -(\beta+\varepsilon_3) \\ 0 & \beta+\varepsilon_3 & \alpha+\varepsilon_2 \end{bmatrix} \quad B_1 = \begin{bmatrix} r & 0 & 0 \\ 0 & p & q \\ 0 & -q & p \end{bmatrix}$$

where, $\mathrm{r} = 1-2\varepsilon_1, p = \alpha - \frac{e}{g}\varepsilon_1 + \varepsilon_2, q = -\beta + \frac{f}{g}\varepsilon_1 - \varepsilon_3$

So the stability of fixed points Y_0^* and Y_1^* is determined by the eigenvalues of matrices B_0 and B_1. At the same time, in order to discuss the existence and stability of invariant cycles at two points, we need to transform the mapping into polar coordinates. Order:

$$\tilde{\varepsilon}_2 = \tilde{\lambda}_2(0)\tilde{\varepsilon}_{20}, \varepsilon_0 = (\varepsilon_1, \varepsilon_{20})^T, \tilde{c} = \tilde{\lambda}_2(0)\tilde{c}_0, \tilde{e} = \tilde{\lambda}_2(0)\tilde{e}_0, \tilde{m} = \tilde{\lambda}_2(0)m_0$$

In polar coordinate form, the original mapping is transformed into

$$\begin{gathered} x' = \varepsilon_1 x + ax^2 + br^2 + gx^3 + hxr^2 + h.o,t \\ r' = r(1+\varepsilon_{20} + c_0 x + e_0 x^2 + m_0 r^2) + h.o.t \\ \theta' = \theta + \theta_0 + \varepsilon_{30} + d_0 x + f_0 x^2 + n_0 r^2 + h.o.t \end{gathered}$$

Where, $\varepsilon_{20} = \alpha\varepsilon_2 + \beta\varepsilon_3, \varepsilon_{30} = \alpha\varepsilon_3 - \beta\varepsilon_3, e_0 = \alpha e + \beta f, f_0 = \alpha f - \beta e, m_0 = \alpha m + \beta n, n_0 = \alpha n - \beta m$

If $\mathrm{a} = \mathrm{b} = c_0 = 0$, we can obtain:

$$\begin{gathered} x' = \varepsilon_1 x + gx^3 + hxr^2 + h.o,t \\ r' = r(1+\varepsilon_{20} + e_0 x^2 + m_0 r^2) + h.o.t \\ \theta' = \theta + \theta_0 + \varepsilon_{30} + d_0 x + f_0 x^2 + n_0 r^2 + h.o.t \end{gathered}$$

The forked bifurcation exists in the mapping. The four fixed points of the mapping are: $A_0 = (0,0), A_1 = \left(0, \sqrt{-\frac{\varepsilon_{20}}{m_0}}\right), A_2 = \left(\sqrt{\frac{-\varepsilon_1}{g}}, 0\right), A_3 = \left(\sqrt{\frac{m_0\varepsilon_1 - h\varepsilon_{20}}{e_0 h - m_0 g}}, \sqrt{\frac{-e_0\varepsilon_1 + g\varepsilon_{20}}{e_0 h - m_0 g}}\right)$

It is easy to obtain:

(i) The local stability conditions of trivial fixed point A_0 are $\varepsilon_1 < 0$ and $\varepsilon_{20} < 0$.

(ii) The existence condition of fixed point A_1 is $\varepsilon_{20}/m_0<0$, and the local stability condition is $\varepsilon_1 < h\varepsilon_{20}/m_0$.

(iii) The existence condition of fixed point A_2 is $\varepsilon_1/g<0$, and the local stability condition is $\varepsilon_1 > 0$.

(iv) The existence conditions of fixed point A_3 are $\frac{m_0\varepsilon_1-h\varepsilon_{20}}{e_0h-m_0g}>0$, $\frac{-e_0\varepsilon_1+g\varepsilon_{20}}{e_0h-m_0g} > 0$, the local stability conditions are $\varepsilon_1 > 0, \varepsilon_{20} > \frac{e_0\varepsilon_1}{g}$.

The topological properties of mappings near fixed points are determined by the coefficients of higher-order terms. Different combinations of coefficients of higher-order terms have different results. We assume that the combinations of coefficients of higher-order terms are as follows:

$$g > 0, m_0 > 0, h < 0, e_0 < 0, e_0h - m_0g > 0$$

The bi-parametric unfolded graph of mapping local dynamics is obtained as follows.

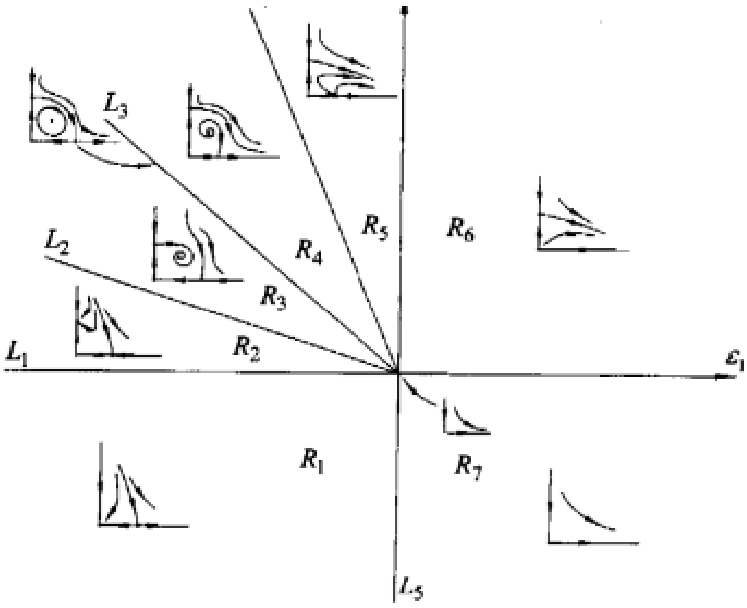

Fig. 2. Two-parameter unfolding diagram of dynamical properties of mapping (11).

The boundary conditions of each region are as follows:

$$L_1:\ \varepsilon_{20} = 0,\ \varepsilon_1 < 0$$

$$L_2:\ \varepsilon_{20} = \frac{m_0}{h}\ \varepsilon_1$$

$$L_3:\ \varepsilon_{20} = \frac{m_0(g - e_0)}{g(h - m_0)}\ \varepsilon_1,\ \varepsilon_1 > 0$$

$$L_4:\ \varepsilon_{20} = \frac{e_0}{g}\ \varepsilon_1,\ \varepsilon_1 > 0$$

$$L_5:\ \varepsilon_1 = 0,\ \varepsilon_{20} < 0$$

As shown in figure2, the fixed points are all periodic 1 fixed points in the regions of $R_1$and R_7, and pitchfork bifurcation will occur when passing through L_5. In the region of R_3 and R_4, the change of fixed point appears invariant circle, and the ring bifurcation will occur when passing through L_2 and L_3. When passing L_4, the doubling periodic bifurcation will occur and the semi-attractive invariant ring will appear.

V. Numerical Simulation and Analysis

Select a set of parameters $\zeta = 0.01, \mu_{c3} = 0.3, \mu_{m3} = 9.5, \mu_{m2} = 1.0, \mu_{k2} = 1.5, \mu_{k3} = 10.0$. The bifurcation parameter is set to frequency ω. When $\omega < 1.15$ The form of motion is single cycle. When ω increases gradually, the motion characteristics of the system change. When $\omega = 1.4$, the stable q = 1/1 fixed point instability occurs and Hopf bifurcation occurs. When $\omega > 1.4$, The system enters chaotic motions.

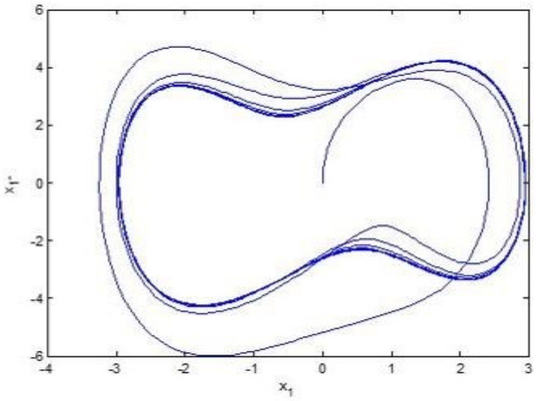

Fig. 3. When $\omega = 1.40$, phase diagram of x_1.

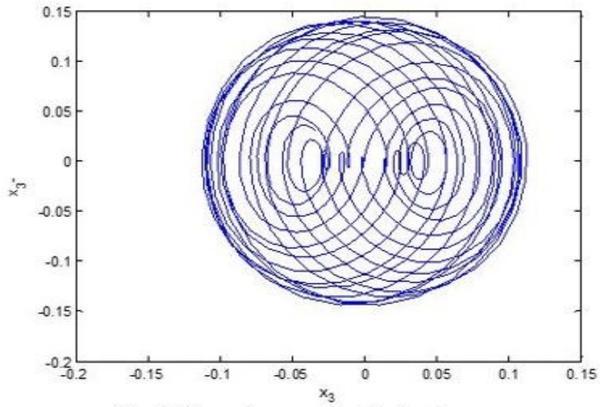

Fig. 4. Phase diagram of x_3 in chaotic case.

VI. Conclusion

In this paper, Using modal analysis and Poincare mapping, three-degree-of-freedom vibro-impact model with clearance was studied. The two-parameter fold conditions of various bifurcations are obtained. The phase diagram of the system is numerically simulated, which provides a theoretical basis for improving the reliability of the mechanical system in engineering practice.

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