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Numerical Schemes for a Class of Elliptic Problems with Non Linear Newton Boundary Conditions

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Abstract **– The error analysis of the finite element approximation of a class of nonlinear elliptic problem is carried out.**

Keywords **– Error Estimate, Finite Element, Elliptic, Nonlinear, Newton Conditions**.

I. Preliminaries

Following the work done in Feistauer [2], we prove the existence and the regularity of the solutions of a class of elliptic problems with nonlinear Newton boundary conditions.

Feistauer suppose that the nonlinear terms have polynomial growth. Such boundary conditions have a number of applications in science and engineering. In particular, this can be made in modelling of electrolysis of aluminum.

The nonlinear boundary conditions describe turbulent flow in boundary layer. In reality the polynomials growth used by Feistauer in [1] is just an approximation of the unknown functions describing the turbulence or boundary layer growth. We consider a large class of functions including asymptotic expansions not necessary polynomials.

We obtain similar results as in Feistauer [1], [2]. The regularity of our solutions is also established in our work that which was not explicit in the Feistauer work. The convergence of the discrete to the continuous solution is proved.

We use the classical theory of nonlinear monotone operator with simple modifications of the proof in Feistauer [1], we obtained an optimal error o(h).

For the linear approximation without numerical approximation integration. The case of numerical integration is also considered. The computation of approximated solution is done by some linearized schemes. Numerical experiments are given, the order of convergence is estimated. In Tchoua and Ita B.I [9] some numerical schemes are proposed and the convergences proved Numerical test are done.

II. Formulation of the Problem

Let Ω be any bounded domain of IR^2 with Lipschitz continuous boundary Γ. Consider the following problem: (P) find $u \in H^1(\Omega)$ satisfying in the distributional sense.

2.1. $\Delta u = f \text{ in } \Omega$

2.2. $((\partial u)/(\partial n)) + \beta(u)\, u \mid \_ \{\Gamma\} = \varphi,\ \beta: IR \to IR$ *are Given Functions*

Where $f \in L^2(\Omega)$; $\varphi \in L^2(\Gamma)$. We first consider the weak solutions of problem (P) defined above in the space $H^1(\Omega)$.

The regularity property is obtained by the Sobolev i, bedding theorem and the classical regularity problems for elliptic problems.

To ensure the solvability of the problem (P) we state some assumptions on the data β.

2.3. *Assumption on β*

Assumption H1

β: IR → IR^ {∗} is strictly increasing function satisfying, β (0) = 0.

Assumption H2

β is locally Lipchitz continuous.

Assumption H3

There exist positive constants α and K > 0, such that for all h >0, inf (((t + h) β (t + h) – tβ (t))/ (h^α)) = K > 0 K is independent of h.

III. Variational Formulation

3.1. *Some Useful Operators*

Define the following forms:

3.1.1. *b (u, v) = ∫ ∇u. ∇vdx for all u, v ∈H^1(Ω)*

3.1.2. *d (u, v) = ∫β (u) uvds*

3.1.2. *L_2 (v) = ∫fvdx*

3.1.3. *a (u, v) = b(u, v) + d(u, v) for all u, v in H^1 (Ω)*

3.2. *Definition*

A function u defined on Ω is a weak solution of problem (P) if and only if

i. u ∈H^1(Ω)

ii. a (u, v) = l(v)   for v in H^1(Ω)

3.3. *Lemma*

If β satisfies the assumptions (H1) and (H2)

Then

A (u, u-v) -a (v, u-v) ≥ $\|u-v\|_1^2$ + c‖u - v‖ ^ {α + 2} 0, ∂Ω Where c and α are constants given in the assumptions (H1), (H2)

Proof

Consider the function g defined by g(t) = tβ(t). The function g is increasing by assumption (H1).

We have : a(u, u-v) – a (v, u-v) = ∫|∇ (u-v) $|^2$dx + ∫[β(u)u - β(v)v] (u-v) ds

Setting q = (u-v) sig (u-v). From assumption (H3) and the fact that g is increasing it follows that [β (u) u-β (v) v] (u-v) ≥ c $|u-v|^{\alpha}$

3.5 *Remark*

The function β (u) = k $|u|^{\alpha}$, α>0, satisfies the assumptions (H1), (H2), (H3). These functions have been considered in Feistauer [2].

3.6 *Theorem*

The form a (u, v) satisfies

i. *The Operator*

v- - →a (u, v) is a continuous form on $H^1(\Omega)$ for a given u in $H^1(\Omega)$

ii. *The Form a is Uniformly Monotone:*

3.6.1. $a(u, u - v) - a(v, u-v) \geq \rho(\|u-v\|_1)$

Where ρ is defined by:

ρ(t) = {

$c_0 t^{\alpha+2}$ si $0 \leq t \leq t_0$

$c_0 t^{\alpha}$ $t > t_0$

c_0, α, are constants appearing in (H1), (H2), (H3)

3.7. $a(u, u) \geq c \|u\|_1^2$ if $\int |u|^2 ds \geq 1$

3.8. $a(u, u) \geq ((c \|u\|^{\alpha+2}) / ((\|\nabla u\|^2 + 1)^{\alpha+1}))$ if $\int |u|^2 ds < 1$

Proof

The results follow from the proof of theorem 1.14 of Feistauer [2] and the proof of theorem 4.2.1 Tchoua and Ita B.I [9].

3.9. *Corollary*

The operator A defined by (Au, v) = a (u, v) is uniformly monotone and coercive.

Proof

(Au-Av, u-v) = a(v, u - v) and the result follows from (3.6.1).

IV. Finite Element Approximation of the Solution of the Problem (P)

We consider a bounded domain Ω with Lipchitz boundary. We assume that Ω is approximated by a family of polygonal domain Ω_h h a positive parameter. We also assume the following regularity for Ω_h

i. Ω_h has all its vertices on Γ the boundary of Ω.

ii. Each Ω_h is subdivided into triangles of length side less or equal to h.

iii. $\Omega = \cup T$.

We assume that the triangularization $\Im_{h}$ satisfies the following properties:

1. Each vertex of Ω_{h} is a vertex of triangles T of $\Im_{h}$
2. Each triangle of $\Im_{h}$ has at least one vertex in the interior of the domain.
3. Two arbitrary triangles T and T' of $\Im_{h}$ are disjoint or have a single vertex in common or a hole side in common.
4. Each triangle of $\Im_{h}$ contains a ball with radius $c_0 h$ and is contained in a ball of radius $c_1 h$ c_0, c_1 are constants independent of h.

4.2. *Discrete Spaces and Interpolations Operators*

The discrete solution is search in the discrete space $V_{h}=\{v_{h}\in C(\Omega)/v_{h}|_{T}$ is linear for all $T\in\Im_{h}\}$.

The following interpolation are established for our discrete spaces and other classical finite element spaces in [4].

Assumption H4

There exist a continuous linear operator :

$\Pi_{h} : H^1(\Omega)\rightarrow V_{h}$ which satisfies

i. $\Pi_{h} (H_0^1 (\Omega)) \subset V_{h0} = V_{h} \cap H_0^1 (\Omega)$

ii. $\|u - \Pi_{h} u\|_{s,\Omega_{h}} \leq ch^{t-s} \|u\|_{t, \Omega}$ $u\in H^{s} (\Omega)$, s = 0, 1, t = 0, 1

iii. $\|u - \Pi_{h} u\|_{s, \Gamma_{h}}, \leq ch^{1/2} \|u\|_{1, 2, \Omega_{h}}$, $u\in H^1 (\Omega)$

4.3. *Discrete Solution of Problem (P)*

A function $u_{h}: \Omega_{h}\rightarrow IR$ is a discrete solution of the problem (P) if

i. $u_{h} \in V_{h}$

ii. $a (u_{h}, v_{h}) = L(v_{h})$

a being the form defined in 4.1

Let (P_{h}) be the discrete problem

4.3. *Theorem*

The discrete problem has a unique solution u_{h}, v_{h},.

Proof

$v_{h}\subset H^1(\Omega)$ and is a closed subspace of $H^1(\Omega)$ (v_{h} is a finite dimensional space) the discrete nonlinear A_{h} associated to the problem (P_{h}) is the restriction of the operator A is defined in 3.9 which is uniformly monotone and coercive by theorem 3.4.

It then follows that A_{h} is monotone and coercive on the finite dimensional space V_{h}. And the result follows.

4.4. *Error Estimates*

Before carrying our error analysis we recall the following result deduced from that of Feistauer [2].

4.4. *Theorem*

Under the assumptions (H1), (H2), (H3) and (H4). The discrete solution u_{h} of ($P_{\{h\}}$) satisfies there exist positive constants c_0, c_1, h_0, h_1 such that: $h_0, h_1<1$

4.5. $\| u - u_{\{h\}} \|_{\{1, \Omega\{h\}\}} \leq \{ch^{\{1-\alpha\}} \|u\| 2, 2, \Omega \leq ch \|u\| 1, 2, \Omega$

$0 < h_0 \leq h_1$

$h_1 \leq h \leq h_0$

u the exact solution of (P).

Proof

Using the assumptions (H3) and (H4) the proof is similar to that of theorem 1.2 in Feist[1], for the special case: $\beta(u) = k\, |u|^{\{\alpha\}}$, $k > 0$, $\alpha > 0$.

4.5. *Lemma*

For each $h \in (0, h_0)$, the discrete solution u_{h} of (P) satisfies: there exists a constant $c > 0$, independent of h such that:

4.6. $\rho_1(\|u - u_{\{h\}}\|_{\{1, 2\}}) \leq c\inf \|u-v_{\{h\}}\|_{\{1, 2\}}$

4.7. $\rho 1\ (t) = ((\rho\ (t)) /t),\ t > 0$

4.8. $\rho(\|u\_\{h\}\|_{\{1, 2\}} \leq c\|u_{\{h\}}\|_{\{1, 2\}}$ *ρ Given in Theorem 3.4*

Proof

From the definition of ρ and theorem 3.4, we have $\rho\ (\|u - u_{\{h\}}\|_{\{1, 2\}}) \leq a\ (u, u - u_{\{h\}}) – a\ (u, u – u_{\{h\}}) = a\ (u, u - u_{\{h\}}) - a\ (u, u - v_{\{h\}})$ for all v_{h}∈V

The function a (u, v) is locally Lipschitz continuous in u. From the relation $\rho(\|u_{\{h\}}\|_{\{1, 2\}}) \leq a\ (u_{\{h\}}, u_{\{h\}}) = \int_\Omega f u_{\{h\}}\, dx + \int_\Omega g u_{\{h\}}\, ds$

And it follows $\rho(\|u_{\{h\}}\|_{\{1, 2\}}) \leq c\ \|u_{\{h\}}\|_{\{1, 2\}}$, and by the definition of ρ we deduce that: $(\|u_{\{h\}}\|_{\{1, 2\}}) \leq c$, and it follows from the relation: $\rho(\|u-u_{\{h\}}\|_{\{1, 2\}}) \leq |a\ (u, u - v_{\{h\}}) – a\ (u, u - v_{\{h\}})| \leq c\ \|u - u_{\{h\}}\|.\| u - u_{\{h\}}\|_{\{1, 2\}}$ for arbitrary $v_{\{h\}}$ in $V_{\{h\}}$ and the result follows.

4.6. *Theorem*

Let us assume (H1), (H2), (H3) and (H4) and that $u \in H^2(\Omega)$, then there exist constants $c>0$, $h_1>0$ Such that: $||u - u_{\{h\}}||_{\{1, 2\}} \leq ch$ for all $0<\theta<1$.

Proof

From lemma 4.5 we have $\int_\Omega |\nabla(u - u_{\{h\}})|^2\, dx \leq c_1 h^2 \|u\|_{\{1, 2\}}^2$

Using the interpolation relations of Banach spaces in $L^P(\Omega)$

Which state : $[L^p, L^q] = L^{p_0}$, $1/P_0 = (1 - \theta)/ p + \theta/q)$ $\theta \in (0, 1)$ $\|u\|_{LP} \leq \|u\|_{LP}{}^{\{1-\theta\}} \|u\|_{Lq}.{}^{\theta}$

The Sobolev imbedding theorems

$H^1(\Omega) \mapsto L^q(\Gamma)$ for $q \geq 1$

$H^1(\Omega) \mapsto L^q(\Omega)$

For $q = \alpha + 2$

$(1/ 2) = ((1 - \theta)/ p) + (\theta/ (\alpha + 2))$ combining, we obtain from the Friedrichs inequality $\|u\|_{\{1, 2, \Omega\}} \leq [\int |\nabla u|^2 dx + \int |u|^2 ds]$ ^ {1/ 2} $\|u - u_{\{h\}}\|_{\{1, 2\}} \leq ch$

4.7 *Commentaries*

The results in theorem 4.6 improve the results in Feistauer [1] and in particular for $\alpha = 1$

1. We obtain the order 2/3 instead of 1/2 as given by theorem 3.3 in Feistauer [2]
2. The results obtained in Feistauer [2] relatively for the use of numerical integration is improved in the same order as in theorem 4.6.

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