International Journal of Innovation in Science and Mathematics
Volume 6, Issue 6, ISSN (Online): 2347–9051

On the Products of k-Fibonacci Numbers and k-Lucas Numbers

Sergio Falcon
Department of Mathematics University of Las Palmas de Gran Canaria (SPAIN).
Corresponding author email id: sergio.falcon@ulpgc.es

Date of publication (dd/mm/yyyy): 10/11/2018

Abstract **– In this paper, we study the relation between the product of two k–Fibonacci numbers, the product of two k–Lucas numbers, and the product of a k-Fibonacci number by a k–Lucas number.**

As a main result, we look for the formula to find the sum of these products.

Then we apply these formulas to some simple cases but more common than general cases and finalize with the generating function of these products.

Keywords **– k–Fibonacci Numbers, k–Lucas Numbers, Binet Identity.**

Academic Discipline and Sub-Disciplines
Mathematics: Combinatorics, Integer sequences, Fibonacci numbers.
Subject Classification: 11B39.

I. Introduction

One of the more studied sequences is the Fibonacci sequence [6, 7], and it had been generalized in many ways [5]. Here, we use the following one-parameter generaliza- -tion of the Fibonacci sequence [3, 4].

1.1. *Definition 1*

For a given integer number $k \geq 1$, the k–Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently [3] as $F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ for $n \geq 1$, with the initial conditions $F_{k,0} = 0,\ F_{k,1} = 1$

First k–Fibonacci numbers are $\{0,1,k,k^2+1,k^3+2k,\dots\}$

Note for k = 1 the classical Fibonacci sequence is obtained and for k = 2 it is the Pell sequence.

From the definition, we can obtain the characteristic equation $r^2 = k\,r + 1$ whose solutions are $\sigma_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \frac{k-\sqrt{k^2+4}}{2}$. These solutions verify the following properties:

$\sigma_1 \cdot \sigma_2 = -1,\ \sigma_1 + \sigma_2 = k,\ \sigma_1 - \sigma_2 = \sqrt{k^2+4}, \sigma^2 = k\sigma + 1,\ \sigma_1 > 0,\ \sigma_2 < 0$

For the properties of the k–Fibonacci numbers, see [3, 4]. In particular, the very used Binet Identity is $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$. Finally, we define the k–Fibonacci numbers of negative index as $F_{k,-n} = (-1)^{n+1} F_{k,n}$.

1.2. *Definition 2*

For a given integer number $k \geq 1$, the k–Lucas sequence, say $\{L_{k,n}\}_{n\in N}$ is defined recurrently [1] as $L_{k,n+1} = k\,L_{k,n} + L_{k,n-1}$ for $n \geq 1$, with the initial conditions $L_{k,0} = 2,\ L_{k,1} = k$

First k–Lucas numbers are $\{2,k,k^2+2,k^3+3k,\dots\}$

The Binet Identity for the k–Lucas numbers is $L_{k,n} = \sigma_1^n + \sigma_2^n$.

The k–Lucas numbers are related to the k–Fibonacci numbers by the relation $L_{k,n} = F_{k,n-1} + F_{k,n+1}$. Moreover, $L_{k,-n} = (-1)^n L_{k,n}$. Other relation between the k–Lucas and the k–Fibonacci numbers is the convolution formula for the k–Lucas numbers (see [2], formula (1)):

$$L_{k,p+q} = F_{k,p} L_{k,q+1} + F_{k,p-1} L_{k,q} \quad (1.1)$$

In [2] the following formulas are proven. For $m, p \in \mathbb{Z}$

$$\sum_{i=0}^{n} F_{k,\,mi+p} = \frac{F_{k,\,m(n+1)+p} - (-1)^m F_{k,\,mn+p} + (-1)^p F_{k,\,m-p} - F_{k,p}}{L_{k,\,m} - (-1)^m - 1} \quad (1.2)$$

$$\sum_{i=0}^{n} (-1)^i F_{k,\,mi+p} = \frac{(-1)^m F_{k,\,m(n+1)+p} + (-1)^{n+m} F_{k,\,m\,n+p} - (-1)^p F_{k,\,m-p} + F_{k,p}}{L_{k,\,m} - (-1)^m - 1} \quad (1.3)$$

$$\sum_{i=0}^{n} L_{k,\,mi+p} = \frac{L_{k,\,m(n+1)+p} - (-1)^m L_{k,\,m\,n+p} + (-1)^p L_{k,\,m-p} - L_{k,p}}{L_{k,\,m} - (-1)^m - 1} \quad (1.4)$$

$$\sum_{i=0}^{n} (-1)^i L_{k,\,m\,i+p} = \frac{(-1)^n L_{k,\,m\,(n+1)+p} + (-1)^{n+m} L_{k,\,m\,n+p} + (-1)^p L_{k,\,m-p} + L_{k,p}}{L_{k,\,m} + (-1)^m + 1} \quad (1.5)$$

II. Product of two k–Fibonacci Numbers and k-Lucas Numbers

First, we prove a lemma that we will need to find the sum of the products of two k–Fibonacci numbers with subscripts in linear form.

2.1 *Theorem: Product of Two k–Fibonacci and k–Lucas Numbers*

For all integer numbers p and q

$$F_{k,p} F_{k,q} = \frac{1}{k^2+4}\left(L_{k,p+q} - (-1)^q L_{k,p-q}\right) \quad (2.1)$$

$$L_{k,p} L_{k,q} = L_{k,p+q} + (-1)^q L_{k,p-q} \quad (2.2)$$

$$F_{k,p} L_{k,q} = F_{k,p+q} + (-1)^q F_{k,p-q} \quad (2.3)$$

Applying the Binet identity, and taking into account $\sigma_1 \cdot \sigma_2 = -1$, we will prove these formulas:

Copyright © 2018 IJISM, All right reserved

(2.1):
$$F_{k,p}F_{k,q}=\frac{\left(\sigma_1^p-\sigma_2^p\right)\left(\sigma_1^q-\sigma_2^q\right)}{k^2+4}=\frac{\left(\sigma_1^{p+q}+\sigma_2^{p+q}\right)-\left(\sigma_1^p\sigma_2^q+\sigma_1^q\sigma_2^p\right)}{k^2+4}=\frac{L_{k,p+q}-\left(\sigma_1^{p-q+q}\sigma_2^q+\sigma_1^q\sigma_2^{p-q+q}\right)}{k^2+4}$$
$$=\frac{L_{k,p+q}-(-1)^q\left(\sigma_1^{p-q}+\sigma_2^{p-q}\right)}{k^2+4}=\frac{L_{k,p+q}-(-1)^qL_{k,p-q}}{k^2+4}$$

(2.2):
$$L_{k,p}L_{k,qw}=\left(\sigma_1^p+\sigma_2^p\right)\left(\sigma_1^q+\sigma_2^q\right)=\left(\sigma_1^{p+q}+\sigma_2^{p+q}\right)+\left(\sigma_1^p\sigma_2^q+\sigma_1^q\sigma_2^p\right)=L_{k,p+q}+\left(\sigma_1^{p-q+q}\sigma_2^q+\sigma_1^q\sigma_2^{p-q+q}\right)$$
$$=L_{k,p+q}+(-1)^q\left(\sigma_1^{p-q}+\sigma_2^{p-q}\right)=L_{k,p+q}+(-1)^qL_{k,p-q}$$

(2.3):
$$F_{k,p}L_{k,q}=\frac{\left(\sigma_1^p-\sigma_2^p\right)\left(\sigma_1^q+\sigma_2^q\right)}{\sigma_1-\sigma_2}=\frac{\left(\sigma_1^{p+q}-\sigma_2^{p+q}\right)+\left(\sigma_1^p\sigma_1^q-\sigma_1^q\sigma_2^p\right)}{\sigma_1-\sigma_2}=F_{k,p+q}+\frac{(-1)^q\left(\sigma_1^{p-q}-\sigma_2^{p-q}\right)}{\sigma_1-\sigma_2}$$
$$=F_{k,p+q}+(-1)^qF_{k,p-q}$$

2.2. *Sum of the Products of two k-Fibonacci Numbers or two k-Lucas Numbers.*

If $p=ai+r$ and $q=bi+s$ in the above equations, then

$$\sum_{i=0}^{n}F_{k,ai+r}F_{k,bi+s}=\frac{1}{k^2+4}\left(\sum_{i=0}^{n}L_{k,(a+b)i+(r+s)}-(-1)^s\sum_{i=0}^{n}(-1)^{bi}L_{k,(a-b)i+(r-s)}\right)\quad(2.4)$$

$$\sum_{i=0}^{n}L_{k,ai+r}L_{k,bi+s}=\sum_{i=0}^{n}L_{k,(a+b)i+(r+s)}+(-1)^s\sum_{i=0}^{n}(-1)^{bi}L_{k,(a-b)i+(r-s)}\quad(2.5)$$

$$\sum_{i=0}^{n}F_{k,ai+r}L_{k,bi+s}=\sum_{i=0}^{n}F_{k,(a+b)i+(r+s)}+(-1)^s\sum_{i=0}^{n}(-1)^{bi}F_{k,(a-b)i+(r-s)}\quad(2.6)$$

where the sums are calculated by using the formulas (1.2), (1.3), (1.4), and (1.5) with and $p=r+s$ or $m=a-b$ and $p=r-s$, as appropriate.

Find the form of these formulas lacks interest and it is much more practical to impose some conditions to the numerical values involved in them. I the following subsections we will apply the formulas (2.4), (2.5), 2nd (2.6) for different values of the coefficients in the subscripts.

2.2.1 *Sum of Squares of the k-Fibonacci Numbers.*

If both subscripts are equals, the formula (2.1) becomes

$$F_{k,p}^2=\frac{1}{k^2+4}\left(L_{k,2p}-2(-1)^p\right)\quad(2.7)$$

Let us suppose s = r = 0 and b = a in the formula (2.4). Then

a) If "a" is odd, the sum of squares with odd subscripts is

$$\sum_{i=0}^{n}F_{k,ai}^2=\frac{1}{k^2+4}\left(\frac{L_{k,2a(n+1)}-L_{k,2an}}{L_{k,2a}-2}-(-1)^n\right)$$

Proof. If we apply the formulas (2.6) and (1.4),

$$\sum_{i=0}^{n}F_{k,ai}^2=\frac{1}{k^2+4}\left(\sum_{i=0}^{n}L_{k,2ai}-2\sum_{i=0}^{n}(-1)^{ai}\right)$$
$$=\frac{1}{k^2+4}\left(\frac{L_{k,2a(n+1)}-L_{k,2an}+L_{k,2a}-L_{k,0}}{L_{k,2a}-2}-2\frac{(-1)^{an}+1}{2}\right)$$
$$=\frac{1}{k^2+4}\left(\frac{L_{k,2a(n+1)}-L_{k,2an}}{L_{k,2a}-2}-(-1)^n\right)$$

b) In similar form, is "a" is even

$$\sum_{i=0}^{n}F_{k,ai}^2=\frac{1}{k^2+4}\left(\frac{L_{k,2a(n+1)}-L_{k,2an}}{L_{k,2a}-2}-2n-1\right)$$

As particular cases, $\sum_{i=0}^{n}F_{k,i}^2=\frac{1}{k^2+4}\left(\frac{L_{k,2n+1}}{k}-(-1)^n\right)$,

$$\sum_{i=0}^{n}F_{k,2i}^2=\frac{1}{k^2+4}\left(\frac{L_{k,4(n+1)}-L_{k,4n}}{L_{k,4}-2}-2n-1\right)$$

If s = r = 1 and b = a, the equation (2.4) becomes

$$\sum_{i=0}^{n}F_{k,ai+1}^2=\frac{1}{k^2+4}\left(\frac{L_{k,2a(n+1)+2}-L_{k,2an+2}+L_{k,2a-2}-L_{k,2}}{L_{k,2a}-2}+2\sum_{i=0}^{n}(-1)^{ai}\right)$$

It is enough to apply the formulas (2.7) and (1.4).

In particular, if a = 2 and taking into account the definition of the k-Lucas numbers, the sum of the squares of the odd k-Fibonacci numbers is

$$\sum_{i=0}^{n}F_{k,2i+1}^2=\frac{1}{k^2+4}\left(\frac{L_{k,4}L_{k,4n+1}+L_{k,3}L_{k,4n}}{k(k^2+4)}+2(n+1)\right)$$

2.2.2 *Sum of the Products of two Consecutive k-Fibonacci Numbers.*

If b = a = 1, r = 0, s = 1 and $\eta=\frac{(-1)^n+1}{2}$, the formula (2.4) is

Copyright © 2018 IJISM, All right reserved

$$\sum_{i=0}^{n} F_{k,i}F_{k,i+1} = \frac{1}{k^2+4}\left(\sum_{i=0}^{n} L_{k,2i+1} - \sum_{i=0}^{n}(-1)^i L_{k,-i}\right) = \frac{1}{k^2+4}$$
$$\left(\frac{L_{k,(2n+1)+1} - L_{k,2n+1} - L_{k,1} - L_{k,1}}{L_{k,2}-2} - L_{k,-1}\eta\right)$$

$$= \frac{1}{k^2+4}\left(\frac{kL_{k,2n+2}-2k}{k^2} - k\eta\right) \to \sum_{i=0}^{n} F_{k,i}F_{k,i+1}$$
$$= \frac{1}{k^2+4}\left(\frac{L_{k,2n+2}-2}{k} - k\eta\right)$$

2.3. *Sum of the Products of two k-Lucas Numbers*

The sum of two k-Lucas numbers is similar to the formula (2.4), without the coefficient $\frac{1}{k^2+4}$ and changing $-(-1)^s$ by $+(-1)^s$; that is

$$\sum_{i=0}^{n} L_{k,ai+r}\, L_{k,bi+s} = \sum_{i=0}^{n} L_{k,(a+b)i+(r+s)} + (-1)^s \sum_{i=0}^{n} (-1)^{bi} L_{k,(a-b)i+(r-s)}$$

In particular:

- $\sum_{i=0}^{n} L_{k,i}L_{k,i+1} = \frac{L_{k,2n+2}-2}{k} + \eta k$
- $\sum_{i=0}^{n} L_{k,i}^2 = \frac{L_{k,2n+1}}{k} + 2 + (-1)^n$

  $\sum_{i=0}^{n} L_{k,2i}^2 = \frac{L_{k,2}L_{k,4n+1} + L_{k,1}L_{k,4n}}{k^3+4k} + 2n + 3 = \frac{L_{k,4n+3} + L_{k,4n+1}}{L_{k,3} + L_{k,1}} + 2n + 3$
- $\sum_{i=0}^{n} L_{k,2i+1}^2 = \frac{L_{k,4}L_{k,4n+1} + L_{k,3}L_{k,4n}}{k^3+4k} - 2n - 3 = \frac{L_{k,4n+5} + L_{k,4n+3}}{L_{k,3} + L_{k,1}} - 2n - 3$

2.4 *Sum of the Products* $F_{k,ai+r}L_{k,bi+s}$

Taking into account the formula (2.1), it is

$$\sum_{i=0}^{n} F_{k,ai+r}L_{k,bi+s} = \sum_{i=0}^{n} F_{k,(a+b)i+(r+s)}$$
$$+(-1)^s \sum_{i=0}^{n} (-1)^{bi} F_{k,(a-b)i+(r-s)}$$

If b = a and s = r, and taking into account $F_{k,ai+r}L_{k,ai+r} = F_{k,2(ai+r)}$:

$$\sum_{i=0}^{n} F_{k,ai+r}\, L_{k,ai+r} = \frac{1}{L_{k,2a}-2}\left(F_{k,2a(n+1)+2r} - F_{k,2an+2r} - F_{k,2a-2r} - F_{k,2r}\right)$$

If b = a and s = r + 1:

$$\sum_{i=0}^{n} F_{k,ai+r}L_{k,ai+r+1} = \frac{1}{L_{k,2a}-2}$$
$$\left(F_{k,2a(n+1)+2r+1} - F_{k,2an+2r+1} - F_{k,2a-2r-1} - F_{k,2r+1}\right) - (-1)^r.$$

In particular, $\sum_{i=0}^{n} F_{k,i}L_{k,i+1} = \frac{F_{k,2n+2}}{k} - \eta$

III. On the Generating Functions

In this section, we will study the generating functions of the different k-Fibonacci and k-Lucas numbers.

It is well known the generating function of the k-Fibonacci numbers is [3] $f(k,x) = \frac{x}{1-kx-x^2}$ and for the k-Lucas numbers is [1] $l(k,x) = \frac{2-kx}{1-kx-x^2}$. By mean of a similar process used in the below papers, we will find the generating functions of the alternated k-Fibonacci numbers and the k-Lucas numbers.

3.1 *Generating Function of the Alternated k-Fibonacci Numbers and k-Lucas Numbers*

From Definition 1,

$$F_{k,2n+1} = kF_{k,2n} + F_{k,2n-1} = (k^2+1)F_{k,2n-1} + k_{k,2n-2} =$$
$$(k^2+2)F_{k,2n-1} - F_{k,2n-3}$$

Let $f_o(k,x)$ be the generating function of the sequence of the odd k-Fibonacci numbers, $\{F_{k,2n+1}\}_{n\in N}$. Taking into account the preceding formula,

$$f_o(k,x) = F_{k,1} + F_{k,3}x + F_{k,5}x^2 + F_{k,7}x^3 + \cdots$$
$$(k^2+2)xf_o(k,x) = (k^2+2)F_{k,1}x + (k^2+2)F_{k,3}x^2 + (k^2+2)F_{k,5}x^3 + \cdots$$
$$x^2 f_o(k,x) = F_{k,1}x^2 + F_{k,3}x^3 + \cdots$$

$$\left(1-(k^2+2)x+x^2\right)f_o(k,x) = F_{k,1} + \left(F_{k,3} - (k^2+2)F_{k,1}\right)x \to f_o(k,x) = \frac{1-x}{1-(k^2+2)x+x^2}$$

In similar form we can find the generating function of the even k-Fibonacci numbers is $f_e(k,x) = \frac{kx}{1-(k^2+2)x+x^2}$.

And for the alternated k-Lucas numbers, these formulas are $l_o(k,x) = \frac{k(1+x)}{1-(k^2+2)x+x^2}$ and $l_1(k,x) = \frac{2-(k^2+2)x}{1-(k^2+2)x+x^2}$, respectively.

3.2 *Generating Function of the Products of two Consecutive k-Fibonacci Numbers and k-Lucas Numbers*

Let $ff(k,x)$ be the generating function of the sequence $\{F_{k,n}F_{k,n+1}\}_{n\in N}$. From the formula (2.1), and taking into account $L_{k,-r} = (-1)^r L_{k,r}$, $F_{k,n}F_{k,n+1} = \frac{1}{k^2+4}\left(L_{k,2n+1} - (-1)^n L_{k,1}\right) = \frac{1}{k^2+4}\left(L_{k,2n+1} - (-1)^n k\right)$. As the generating function of the sequence $\{(-1)^n\}$ is $g(x) = \frac{1}{1+x}$, then

$$ff(x) = \frac{1}{k^2+4}\left(\frac{k(1+x)}{1-(k^2+2)x+x^2} - \frac{k}{1+x}\right) \to ff(x) = \frac{kx}{1-(k^2+1)(x+x^2)+x^3}$$

Similarly, if $ll(k,x)$ is the generating function of the sequence $\{L_{k,n}L_{k,n+1}\}$, from the formula (2.3) it is $ll(k,x) = \frac{k(2-k^2x+2x^2)}{1-(k^2+1)(x+x^2)+x^3}$. Finally, if $fl(k,x)$ is the

Copyright © 2018 IJISM, All right reserved

International Journal of Innovation in Science and Mathematics
Volume 6, Issue 6, ISSN (Online): 2347–9051

generating function of the sequence $\{F_{k,n}L_{k,n}\}$, then $fl(k,x) = f_e(x) = \dfrac{k\,x}{1-(k^2+2)x+x^2}$, because $F_{k,n}L_{k,n} = F_{k,2n}$.

3.3 Generating Function of the Squares of the k-Fibonacci Numbers and the k-Lucas Numbers

Let $f_2(k,x)$ be the generating function of the sequence of squares $\{F_{k,n}^2\}_{n\in N}$. In [3] is proven the formula $F_{k,n+1}^2 = k\,F_{k,2n} + F_{k,n-1}^2$, from where

$$f_2(k,x = F_{k,0}^2 + F_{k,1}^2\,x + F_{k,2}^2\,x^2 + F_{k,3}^2\,x^3 + \cdots \quad \cdot F_{k,1}^2\,x +$$

$$(k\,F_{k,2} + F_{k,0}^2)x^2 + (k\,F_{k,4} + F_{k,1}^2)x^3 + \cdots$$

$$= F_{k,0}^2 + F_{k,1}^2\,x + k\,x(F_{k,2}x + F_{k,4}x^2 + F_{k,6}x^3 + \cdots$$

$$x^2(F_{k,0}^2 + F_{k,1}^2x + F_{k,2}^2x^2 + \cdots \quad x\,f_e(k,x) + x^2 f_2(k,x)$$

$$\to f_2(k,x) = \frac{1}{1-x^2}\left(k + k\,x\frac{k\,x}{1-(k^2+2)x+x^2}\right) = \frac{x(1-x)}{1-(k^2+1)(x+x^2)+x^3}$$

At last, if $l_2(k,x)$ is the generating function of the sequence $\{L_{k,n}^2\}_{n\in N}$ from the formula (2.3) we can get

$$l_2(k,x) = \frac{4-(3k^2+4)x-k^2x^2}{1-(k^2+1)(x+x^2)+x^3}$$

IV. Conclusions

We have found formulas to simplify the products F·F, L·L, and F·L as well as the sums of these products. And we have finalize this investigation with the generating functions of these products in order to simplify the obtaining of the corresponding sequences.

Acknowledgments

Our thanks to the experts who have contributed towards development of the template.

References

[1] S. Falcon 2011, on the k–Lucas numbers, Int. J. Contemp. Math. Sciences, 6(21), 1039–1050.
[2] S. Falcon 2012, on the k–Lucas numbers of arithmetic indexes, Applied Mathematics, 3, 1202–1206. http://dx.doi.org/10.4236/am.2012.310175
[3] S. Falcon, A. Plaza 2007, On the Fibonacci k-numbers, Chaos, Solit. & Fract. 32(5), 1615–24.
[4] S. Falcon, A. Plaza 2007, the k-Fibonacci sequence and the Pascal 2-triangle, Chaos, Solit. & Fract. 33(1), 38–49.
[5] V.E. Hoggat 1969, Fibonacci and Lucas numbers, Palo Alto, CA: Houghton-Mifflin.
[6] A.F. Horadam 1961, a generalized Fibonacci sequence, Math. Mag. 68, 455-9.
[7] S. Vajda 1989, Fibonacci & Lucas Numbers, and the Golden Section, Theory and Applications. Ellis Horwood limited.

Copyright © 2018 IJISM, All right reserved