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Roman Fibonomial Numbers

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***Abstract* – In the beginning of 20th century, the generaliza--tion of binomial coefficient had been proposed by substituting arbitrary sequence in place of natural numbers. Using this, the Fibonomial coefficient had been introduced. In the late 90's binomial coefficients have been generalised in the negative direction, called *Roman binomial coefficient*. In this paper, we introduce *Roman Fibonomial numbers*, which have been divided into six regions. Many interesting properties of these numbers have also been proved using the results related to Fibonacci numbers.**

***Keywords* – Binomial Coefficient, Roman Binomial Coefficient, Fibonacci Numbers, Fibonomial Coefficient.**

AMS Subject Classification Number : 05A10, 11B65, 11B39.

I. Introduction

In 1915, Fontené published a one-page note [4] suggesting a generalization of binomial coefficients, replacing the natural numbers by the terms of an arbitrary sequence $\{A_n\}$ of real or complex numbers. Since 1964, there has been an accelerated interest in the *Fibonomial coefficients* $\begin{bmatrix} m \\ k \end{bmatrix}_F$, which correspond to the choice $A_n = F_n$. In 1949, Dov Jardan [2] introduced this idea by stating more general definition and later considered only Fibonomial case. Fibonomial coefficient have been a popular subject since 1964. One can refer [1], [3], [7], [8], [9] and [10] for further details.

The Fibonomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_F$ is defined for $0 < k \leq n$, by replacing each integer appearing in the numerator and denominator of $\binom{n}{k}$ with its respective Fibonacci number. That is

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{F_n \times F_{n-1} \times \ldots \times F_{n-k+1}}{F_1 \times F_2 \times \ldots \times F_k}.$$

In 1990, Loeb [6] presented a new definition of the factorial which generalizes the usual one and called it "*Roman coefficients*". In the similar manner, we define the generalised approach of Fibonomial coefficient.

We first define the *Roman Fibonomial factorial* F_n^* as

$$F_n^* = \begin{cases} n!_F = F_n \times F_{n-1} \times \ldots \times F_2 \times F_1; n \geq 0 \\ \dfrac{(-1)^{(-n-1)}}{(-n-1)!_F} = \dfrac{(-1)^{(-n-1)}}{F_{-n-1} \times F_{-n-2} \times \ldots \times F_2 \times F_1}; n < 0 \end{cases} \tag{1.1}$$

In table 1, we provide few values of F_n^*.

The following two results follow immediately from the definition:

Lemma 1.1:

For any integer n, $F_n^* \times F_{-n}^* = (-1)^{n-1} F_n$.

Lemma 1.2:

For any integer , $\frac{F_a^*}{F_{a-1}^*} = (-1)^a F_a$.

We next define the *Roman Fibonomial coefficients* $\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F$ for all integers n and k as

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F = \frac{F_n^*}{F_k^* \times F_{n-k}^*} \tag{1.2}$$

Table 2 provides few values of these coefficients.

As we can see in the table, the Roman Fibonomial numbers are divided into six regions depending the values of n and k. We now find the value of Roman Fibonomial numbers of each region in terms of Fibonomial numbers.

II. Properties of Roman Fibonomial Coefficients

Result 2.1:

(The six regions) Let n and k be integers. Depending on what region of the Cartesian plane the point (n, k) is in, the following formulas apply:

Region 1:

If $n \geq k \geq 0$, then obviously $\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F = \begin{bmatrix} n \\ k \end{bmatrix}_F$.

Region 2:

If $k \geq 0 > n$, then by the definition we have $\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F = \frac{F_n^*}{F_k^* \times F_{n-k}^*}$. In this case $n < 0$ and $n < k$ implies that $n - k < 0$. So using lemma 1.1, we get

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F = \frac{(-1)^{(-n-1)} F_{-n}}{F_{-n}^*} \times \frac{1}{F_k^*} \times \frac{F_{k-n}^*}{(-1)^{k-n-1} F_{k-n}}$$
$$= \frac{(-1)^{-k}}{F_k^*} \times \frac{F_{k-n-1}^*}{F_{-n-1}^*}$$
$$= (-1)^k \begin{bmatrix} -n+k-1 \\ k \end{bmatrix}_F; \text{ as } -n+k-1 > 0 \text{ and } k > 0.$$

Region 3:

If $0 > n \geq k$, then using the definition and lemma 1.1, we can write

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_F = \frac{(-1)^{-n-1} F_{-n}}{F_{-n}^*} \times \frac{1}{F_{n-k}^*} \times \frac{F_{-k}^*}{F_{-k} \times (-1)^{-k-1}}; \text{ as } n, k < 0$$
$$= \frac{(-1)^{k-n} \times F_{-k-1}^*}{F_{n-k}^* \times F_{-n-1}^*}$$
$$= (-1)^{n+k} \begin{bmatrix} -k-1 \\ n-k \end{bmatrix}_F; \text{ as } -k-1 > 0 \text{ and } n-k > 0.$$

Using the same techniques, we get the following results:

Region 4:

If $k > n \geq 0$, then

$$\left[\!\left[n \atop k \right]\!\right]_F = (-1)^{n-k+1} \begin{bmatrix} k-1 \\ n \end{bmatrix}_F^{-1}.$$

Region 5:

If $n \geq 0 > k$, then

$$\left[\!\left[n \atop k \right]\!\right]_F = (-1)^{k+1} \begin{bmatrix} n-k \\ -k-1 \end{bmatrix}^{-1}.$$

Region 6:

If $0 > k > n$, then

$$\left[\!\left[n \atop k \right]\!\right]_F = -\frac{1}{F_{k-n}} \begin{bmatrix} -n-1 \\ -k-1 \end{bmatrix}^{-1}.$$

Note that in regions 1, 2 and 3, the Roman Fibonomial coefficients equals Fibonomial coefficients up to permuta--tion and change of sign. In region 4, 5 and 6, the Roman Fibonomial coefficients are expressed simply in terms of the reciprocals of the Fibonomial coefficients.

In particular, the Roman Fibonomial coefficients always equal integers or the reciprocals of integers.

We now derive some of the basic rules for Roman Fibonomial numbers.

Lemma 2.2:

(Complementation rule) For all the integers n and k, $\left[\!\left[n \atop k \right]\!\right]_F = \left[\!\left[n \atop n-k \right]\!\right]_F$.

Lemma 2.3:

(Iterative rule) for all the integers n, k and m, $\left[\!\left[n \atop k \right]\!\right]_F \left[\!\left[k \atop m \right]\!\right]_F = \left[\!\left[n \atop m \right]\!\right]_F \left[\!\left[n-m \atop k-m \right]\!\right]_F$.

These two results follow immediately from the definition.

Lemma 2.4:

(Roman's Identity) For all integers n and k, $\left[\!\left[n \atop k \right]\!\right]_F \left[\!\left[k \atop n \right]\!\right]_F = \frac{(-1)^{n-k-1}}{F_{|n-k|}}$.

Proof:

From the definition,

$$\left[\!\left[n \atop k \right]\!\right]_F \left[\!\left[k \atop n \right]\!\right]_F = \frac{F_n^*}{F_k^* \times F_{n-k}^*} \times \frac{F_k^*}{F_n^* \times F_{k-n}^*} = \frac{1}{F_{n-k}^* \times F_{k-n}^*}.$$

Using lemma 1.2, the result follows.

The following result follows immediately from this lemma.

Corollary 2.5:

$$\left[\!\left[n \atop k \right]\!\right]_F^{-1} = (-1)^{n-k-1} \left[\!\left[k \atop n \right]\!\right]_F \times F_{|n-k|}.$$

Using this, Result 2.1 for regions 4, 5 and 6 can now be expressed as shown below:

Region 4:

If $k > n \geq 0$, then

$$\left[\!\left[n \atop k \right]\!\right]_F = (-1)^{n-k+1} \begin{bmatrix} k-1 \\ n \end{bmatrix}_F^{-1}$$
$$= -F_{|n-k+1|} \begin{bmatrix} n \\ k-1 \end{bmatrix}_F.$$

Region 5:

If $n \geq 0 > k$, then

$$\left[\!\left[n \atop k \right]\!\right]_F = (-1)^{k+1} \begin{bmatrix} n-k \\ -k-1 \end{bmatrix}^{-1}$$
$$= (-1)^{n+k+1} F_{|n+1|} \begin{bmatrix} -k-1 \\ n-k \end{bmatrix}_F.$$

Region 6:

If $0 > k > n$, then

$$\left[\!\left[n \atop k \right]\!\right]_F = -\frac{1}{F_{k-n}} \begin{bmatrix} -n-1 \\ -k-1 \end{bmatrix}^{-1} = (-1)^{-n+k} \begin{bmatrix} -k-1 \\ -n-1 \end{bmatrix}_F.$$

Many results are available for the recurrence relation of Fibonomial numbers [5]. Using the definition of Fibonomial numbers and the basic identity $F_m L_n + F_n L_m = 2F_{m+n}$ relating both Fibonacci numbers and Lucas numbers, we can easily prove the following result.

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = \frac{1}{2}\left(L_k \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + L_{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F \right),$$

where L_n is the n^{th} Lucas number. It is obvious from result 2.1 that this result holds true for the Roman Fibonomial number in region 1. We establish the similar result for other regions too.

Region 2:

$$\left[\!\left[n \atop k \right]\!\right]_F = (-1)^k \begin{bmatrix} -n+k-1 \\ k \end{bmatrix}_F$$
$$= \frac{(-1)^k}{2}\left(L_k \begin{bmatrix} -n+k-2 \\ k \end{bmatrix}_F + L_{-n-1} \begin{bmatrix} -n+k-2 \\ k-1 \end{bmatrix}_F \right)$$
$$= \frac{1}{2}\left(L_k (-1)^k \begin{bmatrix} -n+k-2 \\ k \end{bmatrix}_F + L_{-n-1} (-1)^k \begin{bmatrix} -n+k-2 \\ k-1 \end{bmatrix}_F \right)$$
$$= \frac{1}{2}\left(L_k \left[\!\left[n+1 \atop k \right]\!\right]_F - L_{-n-1} \left[\!\left[n \atop k-1 \right]\!\right]_F \right).$$

Using the similar technique, we can find the result for other regions as follows:

Region 3:

$$\left[\!\left[n \atop k \right]\!\right]_F = \frac{1}{2}\left(L_{n-k} \left[\!\left[n+1 \atop k+1 \right]\!\right]_F - L_{-n-1} \left[\!\left[n \atop k+1 \right]\!\right]_F \right).$$

Region 4:

$$\left[\!\left[n \atop k \right]\!\right]_F = \frac{1}{2}\left(\left(L_{k-1} \times \frac{F_{|n-k+1|}}{F_{|n-k|}} \times \left[\!\left[n-1 \atop k \right]\!\right]_F \right) + L_{n-k+1} \left[\!\left[n-1 \atop k-1 \right]\!\right]_F \right).$$

Region 5:

$$\left[\!\left[n \atop k \right]\!\right]_F = \frac{1}{2}\left(L_{-n-1} \left[\!\left[n \atop k+1 \right]\!\right]_F - \left(L_{n-k} \times \frac{F_{|n+1|}}{F_{|n|}} \times \left[\!\left[n+1 \atop k+1 \right]\!\right]_F \right) \right).$$

Region 6:

$$\left[\!\left[n \atop k \right]\!\right]_F = \frac{1}{2}\left(L_{n-k} \left[\!\left[n+1 \atop k+1 \right]\!\right]_F - L_{-n-1} \left[\!\left[n \atop k+1 \right]\!\right]_F \right).$$

In [5], Gould gives the recurrence formula in the form:

$$\begin{bmatrix} n \\ k \end{bmatrix}_F = F_{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_F + F_{n-k-1} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_F.$$

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Again, from the result 2.1, this formula is obviously true in region 1. We find the similar results for other regions also.

Region 2:

$$\left[\!\left[{n \atop k}\right]\!\right]_F = (-1)^k {-n+k-1 \brack k}_F$$

$$= (-1)^k \left\{F_{k+1} {-n+k-2 \brack k}_F + F_{-n-2} {-n+k-2 \brack k-1}_F\right\}$$

$$= F_{k+1}(-1)^k {-n+k-2 \brack k}_F - F_{-n-2}(-1)^{k-1} {-n+k-2 \brack k-1}_F$$

$$= F_{k+1} \left[\!\left[{n+1 \atop k}\right]\!\right]_F - F_{-n-2} \left[\!\left[{n \atop k-1}\right]\!\right]_F.$$

Using the similar technique, we can find the following formulae.

Region 3:

$$\left[\!\left[{n \atop k}\right]\!\right]_F = F_{n-k+1} \left[\!\left[{n+1 \atop k+1}\right]\!\right]_F - F_{-n-2} \left[\!\left[{n \atop k+1}\right]\!\right]_F.$$

Region 4:

$$\left[\!\left[{n \atop k}\right]\!\right]_F = \left(F_k \times \frac{F_{|n-k+1|}}{F_{|n-k|}} \times \left[\!\left[{n-1 \atop k}\right]\!\right]_F\right) + F_{n-k} \left[\!\left[{n-1 \atop k-1}\right]\!\right]_F.$$

Region 5:

$$\left[\!\left[{n \atop k}\right]\!\right]_F = F_{-n-2} \left[\!\left[{n \atop k+1}\right]\!\right]_F - F_{n-k+1} \times \frac{F_{|n+1|}}{F_{|n|}} \times \left[\!\left[{n+1 \atop k+1}\right]\!\right]_F.$$

Region 6:

$$\left[\!\left[{n \atop k}\right]\!\right]_F = F_{n-k-1} \left[\!\left[{n+1 \atop k+1}\right]\!\right]_F - F_{-n} \left[\!\left[{n \atop k+1}\right]\!\right]_F.$$

The following result holds true from the definition of Fibonomial numbers.

$$F_{n-k} {n \brack k}_F = F_n {n-1 \brack k}_F.$$

Again, from the result 2.1, this formula is obviously true in region 1. We find the similar results for other regions.

Region 2:

$$\left[\!\left[{n \atop k}\right]\!\right]_F = (-1)^k {-n+k-1 \brack k}_F$$

i.e. $F_{-n-1} \left[\!\left[{n \atop k}\right]\!\right]_F = (-1)^k F_{-n-1} {-n+k-1 \brack k}_F$

$$= (-1)^k F_{-n+k-1} {-n+k-2 \brack k}_F$$

$$= F_{-n+k-1} \left[\!\left[{n+1 \atop k}\right]\!\right]_F.$$

Using the similar technique, we can obtain the result for other regions as follows:

Region 3:

$$F_{-n-1} \left[\!\left[{n \atop k}\right]\!\right]_F = F_{-k-1} \left[\!\left[{n+1 \atop k+1}\right]\!\right]_F.$$

Region 4:

$$F_{n-k} \left[\!\left[{n \atop k}\right]\!\right]_F = F_n \left[\!\left[{n-1 \atop k}\right]\!\right]_F.$$

Region 5

$$: F_{n+2} \left[\!\left[{n \atop k}\right]\!\right]_F = (-1)^{n+1} F_{-k-1} \left[\!\left[{n+1 \atop k+1}\right]\!\right]_F.$$

Region 6:

$$F_{n-k} \left[\!\left[{n \atop k}\right]\!\right]_F = -F_{-k-1} \left[\!\left[{n \atop k+1}\right]\!\right]_F.$$

In this paper, we have introduced the Roman Fibonomial numbers and discussed some of its results. More work related to them has potential and under progress.

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Table 1.

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
F_n^*	$-\frac{1}{30}$	$\frac{1}{6}$	$-\frac{1}{2}$	1	-1	1	1	1	1	2	6	30	240

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Table 2.

$n\backslash k$	-5	-4	-3	-2	-1	0	1	2	3	4	5
5	1/680680	-1/37128	1/2184	-1/104	1/8	1	5	15	15	5	1
4	1/61880	-1/5460	1/520	-1/40	1/5	1	3	6	3	1	1/5
3	1/5460	-1/780	1/120	-1/15	1/3	1	2	2	1	1/3	-1/15
2	1/520	-1/120	1/30	-1/6	½	1	1	1	½	-1/6	1/30
1	1/40	-1/15	1/6	-1/2	1	1	1	1	-1/2	1/6	-1/15
0	1/5	1/3	½	-1	1	1	1	-1	½	1/3	1/5
-1	1	-1	1	-1	1	1	1	1	-1	1	-1
-2	-3	2	-1	1	-1	1	-1	2	-3	5	8
-3	6	-2	1	1	-1	1	-2	6	-15	40	-104
-4	-3	1	-1/2	-1/2	-1/2	1	-3	15	-60	260	-1092
-5	1	-1/3	-1/6	1/6	-1/3	1	-5	40	-260	1820	-12376