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A Study of W_8-Curvature Tensor on Generalized Sasakian Space Forms

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Abstract **– In this paper, we study flatness properties of the W_8 - curvature tensor on generalized Sasakian space form.**

Keywords **– W_8 - Curvature Tensor, Study Flatness Properties, Generalized Sasakian Space Forms.**

I. INTRODUCTION

A Sasakian manifold $M(\varphi,\xi,\eta,g)$, is said to be a Sasakian-space form if all the φ- sectional curvatures $K(X \,\text{Ù}\, \varphi X)$ are equal to a constant c, where $K(X \,\text{Ù}\, \varphi X)$ denotes the sectional curvature of the section spanned by the unit vector field X, orthogonal to ξ and φX. In such a case, the Riemannian curvature tensor of M is given by [1].

$$R(X,Y)Z = \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}$$
$$+\frac{c-1}{4}\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \tag{1.1}$$

These spaces can be modeled depending on $c > -3$, $c = -3$, $c < -3$.

As a natural generalization of these manifolds, Alegre, Blair and Carriazo introduced and studied the notion of generalized Sasakian space forms in 2004 [1]. They replaced constant quantities $\frac{c+3}{4}$ and $\frac{c-1}{4}$ of relation (1.1) by differentiable functions f_1, f_2, *and* f_3.

An almost contact metric manifold $M(\varphi,\xi,\eta,g)$ is said to be a generalized Sasakian space –form if the curvature tensor R is given by [1].

$$R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\}$$
$$+f_3\{g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} \tag{1.2}$$

where f_1, f_2, and f_3 are differentiable functions on M and X,Y,Z are vector fields on M. In such a case the manifold is denoted by $M(f, f_2, f_3)$.

II. PRELIMINARIES

In an almost contact metric manifold $M^{2n+1}(\varphi,\xi,\eta,g)$, where φ is a (1, 1) tensor field, ξ is a contravariant vector field, η is a 1-form and g is the compatible Riemannian metric, we have [3]

$$\varphi^2 X = -X + \eta(X)\xi,\ \ \eta(\xi) = 1,\ \ \varphi\xi = 0,\ \ \eta(\varphi X) = 0, \tag{2.1}$$

$$g(X,\xi)=\eta(X), \tag{2.2}$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.3}$$

$$g(\varphi X,X)=0 \tag{2.4}$$

$$g(\varphi X,Y)=-g(X,\varphi Y), \tag{2.5}$$

$$\left(\tilde{N}_X\eta\right)(Y)=g(\tilde{N}_X\xi,Y) \tag{2.6}$$

In a $(2n+1)$- dimensional generalized Sasakian space-form the following relations hold

$$\eta(R(X,\ Y)\ Z)\ =\ (f_1-f_3)\ \{g(Y,Z)\ \eta\ (X)-g\ (X,\ Z)\ \eta\ (Y)\} \qquad (2.5)$$

$$\eta(R(X,Y)\xi)=0 \qquad (2.6)$$

$$\eta\left(R(\xi,X)Y\right)=\left(f_1-f_3\right)\left\{g(X,Y)-\eta(X)\eta(Y)\right\} \qquad (2.7)$$

$$R(X,Y)\xi=\left(f_1-f_3\right)\left\{\eta(Y)X-\eta(X)Y\right\} \qquad (2.8)$$

$$R(\xi,Y)Z=\left(f_1-f_3\right)\left\{g(Y,Z)\xi-\eta(Z)Y\right\} \qquad (2.9)$$

$$g\left(R(\xi,X)Y,\xi\right)=\left(f_1-f_3\right)\left\{g(\varphi X,\varphi Y)\right\} \qquad (2.10)$$

$$R(\xi,X)\xi=\left(f_1-f_3\right)\varphi^2X \qquad (2.11)$$

$$R(\xi,X)\xi=\left(f_1-f_3\right)\left\{\eta(X)\xi-X\right\} \qquad (2.12)$$

$$S(X,Y)=\left(2nf_1+3f_2-f_3\right)g(X,Y)-\left(3f_2+(2n-1)f_3\right)\eta(X)\eta(Y) \qquad (2.13)$$

$$S(X,\xi)=2n(f_1-f_3)\eta(X) \qquad (2.14)$$

$$S(\varphi X,\varphi Y)=S(X,Y)-2n(f_1-f_3)\eta(X)\eta(Y) \qquad (2.15)$$

$$r=2n(2n+1)f_1+6nf_2-4nf_3 \qquad (2.16)$$

$$QX=(2nf_1+3f_2-f_3)X-(3f_2+(2n+1)f_3)\eta(X)\xi \qquad (2.17)$$

$$Q\xi=2n(f_1-f_3) \qquad (2.18)$$

Where Q is the Ricci operator, that is, $g(QX,Y)=S(X,Y)$

Here, S is the Ricci tensor and r is the scalar curvature of the space form.

It is well known that a generalized Sasakian space form of dimension $(2n+1)$ with condition $(n>1)$ is η-*Einstein* space -form if its Ricci tensor S satisfies the condition;

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y) \qquad (2.19)$$

For arbitrary vector fields X and Y, where a and b are smooth functions on M.

From (2.13) we have

$$a = \left(2nf_1 + 3f_2 - f_3\right) \text{ and } b = -\left\{3f_2 + (2n-1)f_3\right\} \tag{2.20}$$

Also from (1.1) we get

$$R(X,Y)\xi = \left(f_1 - f_3\right)\left[\eta(Y)X - \eta(X)Y\right] \tag{2.21}$$

Where $\left(\eta(Y)X - \eta(X)Y\right) = \pi(X,Y)$ known as the torsion tensor which is non-zero for spaces admitting semisymmetric metric connections. This therefore, follows that for a flat manifold $\left(f_1 - f_3\right) = 0$

III. W_8 - Curvature Tensor in a Generalized-Sasakian Space-Form

Mishra and pokhariyal [4] gave the definition of W_8 -curvature tensor as

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right] \tag{3.1}$$

3.1. *Definition*

A generalized Sasakian space form M of dimension $(2n+1)$ is said to be W_8 -flat [5] if W_8 -curvature tensor vanishes identically, implying $W_8(X,Y)Z = 0$.

3.2. *Theorem*

If a $(2n+1)$- dimensional generalized Sasakian space-form $M(f, f_2, f_3)$ is W_8 - flat, then $f_3 = f_1 = \frac{3f_2}{1-2n}$

3.3. *Definition*

A generalized Sasakian space form M of dimension $(2n+1)$ is said to be flat if the Riemannian-curvature tensor vanishes identically, that is $R(X,Y)Z = 0$.

Proof.

If generalized Sasakian space form is W_8 - flat, then $W_8(X,Y)Z = 0$

Therefore equation (3.1) becomes,

$$0 = R(X,Y)Z + \frac{1}{n-1}\left[S(X,Y)Z - S(Y,Z)X\right] \tag{3.2}$$

Taking inner product of equation (3.2) with V we get,

$$0 = g\left(R(X,Y)Z,V\right) + \frac{1}{n-1}\left[g\left(S(X,Y)Z,V\right) - g\left(S(Y,Z)X,V\right)\right] = g\left(R(X,Y)Z,V\right) + \frac{1}{n-1}\left[\left(S(X,Y)g(Z,V)\right) - \left(S(Y,Z)g(X,V)\right)\right] \tag{3.3}$$

Putting $V = \xi$ in (3.3) we get

$$0 = \eta\left(R(X,Y)Z\right) + \frac{1}{n-1}\left[\left(S(X,Y)\eta(Z)\right) - \left(S(Y,Z)\eta(X)\right)\right] \tag{3.3}$$

Using equations (2.5) and (2.13) in equation (3.3) gives

$$0 = f_1\{g(Y,Z)\eta(X) - g(X,Z)(Y)\} + f_3\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\} + \frac{1}{n-1}\left[a\{g(X,Y)\eta(Z) - g(Y,Z)\eta(X)\} + b\{\eta(X)\eta(Y)\eta(Z) - \eta(Y)\eta(Z)\eta(X)\}\right] \quad (3.4)$$

Simplifying equation (3.4) yields

$$0 = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + \frac{1}{n-1}\left[a\{g(X,Y)\eta(Z) - g(Y,Z)\eta(X)\}\right] \quad (3.5)$$

Putting $Y = \xi$ in (3.5) gives

$$0 = (f_1 - f_3)(\eta(Z)\eta(X) - g(X,Z)\xi) \quad (3.6)$$

Since $(\eta(Z)\eta(X) - g(X,Z)\xi) \neq 0$ for a generalized Sasakian space-form, it implies that,

$$f_1 = f_3 \quad (3.7)$$

Again, if instead we put $Z = \xi$ in (3.5) for a W_8 -flat generalized Sasakian space form, we shall have

$$0 = (f_1 - f_3)\{\eta(Y)\eta(X) - \eta(X)\eta(Y)\} + \frac{1}{n-1}\left[a\{g(X,Y)\xi - \eta(Y)\eta(X)\}\right] \quad (3.8)$$

which reduces equation (3.8) to equation

$$0 = \left[a\{g(X,Y)\xi - \eta(Y)\eta(X)\}\right] \quad (3.9)$$

Using equation (2.13) in equation (3.9) yields

$$0 = (2nf_1 + 3f_2 - f_3)(g(X,Y)\xi - \eta(Y)\eta(X)) \quad (3.9)$$

Since $(g(X,Y)\xi - \eta(Y)\eta(X)) \neq 0$ for a generalized Sasakian space- form, it implies that, $0 = (2nf_1 + 3f_2 - f_3)$

$$\Rightarrow f_3 = (2nf_1 + 3f_2) \quad (3.10)$$

Putting equation (3.7) into equation (3.10) gives

$$(1-2n)\, f_3 = 3f_2 \Rightarrow f_1 = f_3 = \frac{3f_2}{1-2n} \quad (3.11)$$

This completes the proof of the theorem.

3.4. *Theorem*

A W_8-flat generalized Sasakian space-form is a flat manifold.

Proof

From equation (2.8), $R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y]$

It is clear that when the generalized Sasakian space form is W_8-flat then relation (3.7) reduces equation (2.8) to

$$R(X,Y)\xi = (0)[\eta(Y)X - \eta(X)Y]$$
$$R(X,Y)\xi = 0 \Rightarrow R(X,Y)Z = 0 \quad (3.10)$$

Thus, the theorem.

3.5. *Definition*

A generalized Sasakian space-form $M(f_1, f_2, f_3)$ of dimension $(2n+1)$ is said to be ξ - W_8 - flat [2] if

$$W_8(X,Y)\xi = 0 \tag{3.11}$$

3.6. *Theorem*

If a $(2n+1)$ –dimensional generalized Sasakian space-form $M(f_1, f_2, f_3)$ is ξ - W_8 - flat, then $f_3 = 2nf_1 + 3f_2$.

Proof.

Suppose the condition $W_8(X,Y)\xi = 0$ holds in a $(2n+1)$ –dimensional generalized Sasakian space-form. Then using equation (2.13) in equation (3.1) yields

$$W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}\left[a\{g(X,Y)Z - g(Y,Z)X\} + b\{\eta(X)\eta(Y)Z - \eta(Y)\eta(Z)X\}\right] \tag{3.12}$$

Putting $Z = \xi$ in (3.12) we get,

$$W_8(X,Y)\xi = R(X,Y)\xi + \frac{1}{n-1}\left[a\{g(X,Y)\xi - \eta(Y)X\} + b\{\eta(X)\eta(Y)\xi - \eta(Y)X\}\right] \tag{3.13}$$

Performing an inner product on equation (3.13) and using equation (2.6) yields

$$\eta\left(W_8(X,Y)\xi\right) = 0 + \frac{1}{n-1}\left[a\left(g(X,Y) - \eta(Y)\eta(X)\right) + b\left(\eta(X)\eta(Y) - \eta(Y)\eta(X)\right)\right] \tag{3.14}$$

Which reduces to

$$\eta\left(W_8(X,Y)\xi\right) = \frac{1}{n-1}\left[a\left(g(X,Y) - \eta(Y)\eta(X)\right)\right] \tag{3.15}$$

Since $W_8(X,Y)\xi = 0$, then equation (3.15) reduces to

$$0 = a\{g(X,Y) - \eta(Y)\eta(X)\} \tag{3.16}$$

But a generalized Sasakian space -form has $\{g(X,Y) - \eta(Y)\eta(X)\} \neq 0$, which implies that for equation (3.16) to hold $a = 0$.

From equation (2.20), relation (3.16) will therefore reduce to

$$\begin{aligned} &a = 2nf_1 + 3f_2 - f_3 = 0 \\ &\Rightarrow f_3 = 2nf_1 + 3f_2 \end{aligned} \tag{3.17}$$

Hence, the proof of the theorem.

3.3. *Corollary*

A W_8-flat $(2n+1)$ –dimensional generalized Sasakian space-form $M(f_1, f_2, f_3)$ has a vanishing scalar curvature tensor $(r = 0)$.

Proof.

From (3.11) it is clear that for W_8 - flat $(2n+1)$ –dimensional generalized Sasakian space-form $M(f_1, f_2, f_3)$,

$$f_1 = f_3 = \frac{3f_2}{1-2n}$$

Given the scalar curvature tensor from equation (2.16) as $r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3$ and using equation (3.9) in equation (2.16) we get

$$r = 2n\left(2n+1\right) f_1 + \frac{6n(1-2n)f_1}{3} - 4nf_1$$
$$r = f_1\left[4n^2 + 2n + 2n - 4n^2 - 4n\right] \qquad (3.16)$$
$$r = 0$$

Thus, the theorem.

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