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Sets Theory and Antinomies: a Note

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***Abstract* – In Frege Set Theory the axiom of unlimited abstraction causes various antinomies, revealing a lack of congruence of the same theory. Some solutions to this problem have been advanced by Russell, Whitehead and Zermelo. In this note it is shown a way to abolish the assumption that a set can be a subset of itself, so allowing us to solve various antinomies in a simpler way.**

***Keywords* – Set Theory, Improper Subsets, Antinomies, Grelling's Paradox, Liar Paradox.**

I. Introduction

Antinomies in Set Theory

In Frege Set Theory the axiom of unlimited abstraction causes some antinomies, so revealing a lack of congruence of the same theory (Smullyan, 1992).

This lack of congruence is the object of a large debate.

Russell and Whitehead (1910/1912/1913) try to give a quite complex solution to this problem, but a simpler way is followed by Zermelo (Smullyan, 1992), abandoning the axiom of unlimited set abstraction.

Godel (1931) emphasizes the self-referentiality as cause of antinomies. A significant contribute to the debate is offered by Tarsky (1935, 1944). Recently a new logic, the Fuzzy logic, proposes an interesting way to solve the problem of antinomies (among others: Kosko, 1993).

In this note it will be shown a way to definitely abolish the assumption that a set can be an (improper) subset of itself.

We are ashamed to propose such a hypothesis and therefore we apologize for our daring but we also realize that it allows us to solve with great simplicity various antinomies.

II. Why a Set can not be an (Improper) Subset of Itself

When a subset has the same elements of the set in which it is, it is named *improper set*. We know also that every set is equal to itself (for example, given a set A, we say that $A = A$).

If we have a set A which contains itself, then it will be $A = A$ and $A \in A$.

If A can be a subset of itself, then among its other elements (or, if absent other elements, besides the empty subset), there will be A again. Let us name A_1 the set A and A_2 the subset. Also A_2 can be subset of itself. We will name A_3 the new subset and so on, endlessly.

It is important to note that A_1 is not equal to A_2 and A_2 is not equal to A_3, and so on. Each subset will seem equal to the set which contains it, but it will not be equal to it. In fact, given a set A_i, the infinite sequence of sets and subsets which begins in the set A_i (A_i, A_{i+1}, A_{i+2}, …) can be put into biunivocal correspondence with any of the infinite sequences which begin in its subsets A_{i+1}, A_{1+2}, A_{1+3} ... , but all sequences that begins in A_{i+x}, where x is a positive integer, can not be put into biunivocal correspondence with the sequence which begins in A_i, because while being infinite as the last one, they lack one or more initial elements.

Consequently, if we admit that A can be a member of itself, we must admit that it is not equal to itself too. Then there must be a slight difference in A_2 so that it can be inserted in A_1 without violating the principle that everything is equal to itself.

III. Some Examples

In the Grelling's paradox (1907) we have two sets: *auto-logical* and *hetero-logical adjectives*. Also *auto-logical* and *hetero-logical* are adjectives, but for definition they are more precisely *adjectives of adjectives* (we could name them *second degree adjectives*) and therefore they must be outside of the sets of simple adjectives. Even if we want to place them into the sets of *auto-logical* and/or *hetero-logical adjectives*, there is however a big difference between the adjective *auto-logical* (or *hetero-logical*) and the set of *auto-logical* (or *hetero-logical*) *adjectives*,.

"This statement is false". "In this statements there is three errors".

And again: Socrates: "What Plato says is true". Plato: "What Socrates told is false".

In algebra a function is a relation between the elements of two sets: for each element of a set A there is only an element of a set B. In a function an element does never depend on itself. The dependent variable could, in a certain sense, depend on itself, but only by means of its derivative, into a differential equation with time as independent variable.

Also a statement expresses a relation between the elements of two sets A and B. The statements, in their turn, are the elements of a set, say it S_1. The statements can be correct or incorrect (even the presence of an element in set A or B can be incorrect). No statement can usefully express a relation between an element and itself: the logic of a statement is the same of a function. So the first and the second example at the first line of this paragraph are simply incorrect statements.

About the example at the second line of this paragraph, the statements of a subject (in the example Socrates) are then a set S_1. Their truth or falsity can be verified by another subject (in the example Plato) or by Socrates himself, but in a later moment, in which he has different information and/or judgment methodologies (thus prefiguring an analogy with the differential equations).

The (Plato's) statements about the veracity of Socrates' statements are therefore in a set S_2 which contains S_1.

If the statements of Socrates are in a set S_1 that can be examined only into a greater set S_2, they, in their turn, can not verify statements which are in S_2, but outside of S_1. To do it S_1 should contain S_2, but it is S_2 which contains S_1 and not vice versa.

IV. Conclusions

In this paper we maintain that a set can not be a subset of itself.

The abolition of improper sets allows to simplify the set theory and, above all, to solve many antinomies in a very simple way. As we have shown in the examples, some antinomies can be seen as incorrect statements, so recalling the solution already proposed in third century B.C. by Crisippo (Nobile), others (for example the Grelling's paradox), simply disappear.

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