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Numerical Solutions of a Class of Nonlinear Ordinary Differential Equations by the Differential Transform and Adoman Methods

Paul Tchoua[1*]
[1]Department of Mathematics and Computer Science, University of Ngaoundere, Cameroon

Benedict I. Ita[2]
[2]Department of Pure and Applied Chemistry, University of Calabar, Calabar Cross River State, Nigeria

*Corresponding author email id: tchoua_paul@yahoo.fr

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***Abstract* — In this paper Differential Transform Method (DTM) and Adomian Decomposition Method are used to solve a class of nonlinear differential equations of second order. This method can be applied to many types of liner and nonlinear ordinary differential equations to solve approximately and in some cases give the exact analytical solutions. It reduces the size of computational work while still providing the solutions in terms of series with the convergence rate. Some examples are also given to buttress our points.**

***Keywords* — Differential Transform Method, Nonlinear differential equations of second order, Analytic solutions -Adomian Decomposition Method**

I. Introduction

The nonlinear differential equations are not in general easy to tackle and to be solved analytically. They are generally solved by numerical approximating procedures. The solutions are just known approximately.

A class of new methods is now developed which can enable us to have analytical solutions or any solution for an arbitrary order of accuracy of some nonlinear problem. The differential transform method and the reduced differential transform method are recent efficient methods that can be applied in solving nonlinear differential equations.

These equations arise in the modeling of different natural phenomena in biology, fluid mechanics, chemistry and so on. These equations can be solved numerically with some level of accuracy, but it is always very difficult to obtain analytical solutions. It is therefore very important to develop very efficient schemes capable of giving very accurate solutions of such nonlinear problem. In the literature one can find some numerical techniques with this aim such as wavelet-galerkin- method (WGM); lagrange - interpolation method (LIM); Adomain Decomposition Method (ADM).

Taylor Polynomial Method, Homotopy perturbation Method. Most of these methods are specific to some classes of equations and as such no general methodology can be found in computing the solutions.

We consider the following class of ordinary differential equations of the second order:

$$\frac{d^2u}{dt^2}+F(U)=0 \qquad (1)$$

$$U(0)=a;U(0)=\beta \qquad (2)$$

The initial value problem(1) - (2) is labeled problem (P). We note that many useful mathematical laws in natural sciences are in this form. In equation (1) F is analytic in an open domain D of R containing the origin 0 and satisfying

$$F(0)=0 \qquad (3)$$

We use the differential transform method to give approximate solutions to the problem (P). we consider for applications the class of function

$$F(u)=\frac{u^k}{(1+u^2)^{\alpha}} \qquad (4)$$

k being an integer □ a positive real. Equation (4) shall be implemented for $k = 1,2,3$ and

$$\alpha=1;\ \alpha=\frac{1}{2}\text{with } D=\{x\in R/|x|<1\} \qquad (5)$$

We shall prove that for this particular case there exists a global and unique solution which remains entirely in D if $U(0)$ is in D.

We use the following analytical expansion in D.

$$\frac{u^k}{(1+u^2)^{\alpha}}=u^k\left(\sum_{p=0}^{\infty}(-1)^p u^{2p}\right) \quad \alpha=1 \qquad (6)$$

For the solution out of D not its boundary we set

$$v=\frac{1}{u},|u|>1$$

and apply the expansion (6) to get u

$$g(v)=\frac{v^{2-k}}{1+v^2} \qquad (7)$$

Remark

Let us mention that the problem could be solved without using the expansion (6) which is even easier but we must find by using a software of symbolic calculation evaluate$\frac{d^kF(U)}{dt^k}$ it is sufficient to set $u(t)=\sum_{p=0}^{N}a_pt^p$for truncatedvalue of order N. We have used this approach in our calculation using the software MATLAB. Let us now present the main ideas of the differential transform method (DTM).

2. Element on the Differential Transform Method

Following the work in [1] and [2] the initial function $u(t)$ is supposed analytic in the domain D. We defined the differential transform at point to t_0 be $U(k)$ or some time just denoted Uk.

By

$$U_k=\frac{1}{k!}\left[\frac{d^ku(t)}{dt^k}\right]_{t=t_0} \qquad (8)$$

The following properties can be easily computed from the definition

$$P1: w(t) = u(t)\cdot v(t)$$
$$W(k) = \sum_{l=0}^{k} U_{l-}V_{k-1}$$
$$P2: w(t) = \alpha u(t)\cdot \beta v(t)$$
$$W(k) = \alpha U_k + \beta V_k$$
$$P3: w(t) = \int_0^1 v_1(s)v_2(s)ds$$
$$W(k) = \frac{1}{k}\sum_{l=0}^{k-1} V_{1l}V_{2k-l-1}$$
$$P4: w(t) = u^m(t)$$

$W_k = \sum_{l=0}^{k} U(I)U^{m-1}(k-1)$This can be treated U^p being the notation for the (DTM) of $u^p(t)$. The inverse formula of the (DTM) of $W(k)$ is given by

$$u(t) = \sum_{k=0}^{\infty} U_k t^k \qquad (9)$$

here we have taken $t_o = 0$.
For the details see [6]; [7] and [8].

3. Differential Transform Of Equation (1)

The equation (1) can be transformed into the following equivalent integral equation.

$$\frac{du}{dt} = \int_0^t F(u)(s)ds = \beta \qquad (10)$$

B being given by the initial condition (2). We use the following elementary calculations to compute U_k

$$\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = -\frac{x^2-1}{(1+x^2)^2} = F(x); F(0) = 1$$

$$\frac{d}{dx}\left(\frac{x^2-1}{(1+x^2)^2}\right) = -2\frac{x}{(1+x^2)^3}(x^2-3)$$
$$= -2\cdot 0x\frac{x^2-3\cdot 0}{(x^2-1\cdot 0)} = -F'(x); F'(0)$$
$$= 0$$
$$\frac{d}{dt}\left(-2\frac{x}{(1+x^2)^3}(x^2-3)\right) = \frac{6}{(1+x^2)^4}(x^4-6x^2+1)$$
$$= -F''(x); F''(0) = -6$$

and using the formula from equation (1)

$$\frac{d^2u}{dt^2} + F(U) = 0$$

we have the following

$$\frac{d^2u}{dt^2} = -F(U)(11a)$$
$$\frac{d^2u}{dt^2} = -F(u)\frac{du}{dt}(11b)$$

$$\frac{d^2u}{dt^2} = -F'(u)\left(\frac{du}{dt}\right)^2 - F(u)\frac{d^2u}{dt^2}$$

It follows from the assumption (3) that

$$\frac{d^2u}{dt^2}(0) = 0; \frac{d^2u}{dt^2}(0) = -\alpha; \frac{d^2u(0)}{dt^2} = 0$$

We can then deduce the following values

$$U(0) = \alpha \qquad (12a)$$
$$U(1) = \beta \qquad (12b)$$
$$U(2) = 0 \qquad (12c)$$
$$U(3) = \frac{-\alpha}{6} \qquad (12d)$$
$$U(4) = 0 \qquad (12e)$$

It follows from these calculations for $\alpha = 0$ that the unique stationary solution is also zero by the (DTM) and all the coefficients being zero. We suppose now that the solution is not trivial i.e. $\alpha \neq 0$. Note also that if $\beta \neq 0$ for $\alpha = 0$ the problem has no solution taking $\alpha \neq 0$, then

$$U(0) = \alpha$$
$$U(1) = \beta$$
$$U(2) = -F(\alpha)/2$$
$$U(3) = -F(\alpha)\beta/6$$
$$U(4) = -F'(\alpha)\big(U(1)\big)^2 - 2F(\alpha)U(2)$$

General differential transform for the solution $u(t)$:

Using the relation given by properties P3 and relation (6)

$$(k-1)U_{k-1} + \frac{1}{4}\sum_{I=0}^{k-1} U(1)U(k-I-1) = 0 \qquad (13)$$

using some finite terms in p yield some approximation of the U_k . Let us choose p = 1, we have two terms obtained as

$$(k-1)U_{k-1} + \frac{1}{4}U_k - \sum_{I=0}^{k-1} U(1)U(k-I-1) = 0 \qquad (14)$$

$$k = 1\cdot U_2 = \alpha^2 - \frac{\beta}{2}$$

We can note here that even for k = 1, the approximation is good at first order if p = 0, otherwise the approximation is not good. We can then evaluate the U_k for k = 2, 3, 4 ……..

The values given in (14) are the exact solution. Equation (6) gives approximate solution since the function F(u) or g(v) are approximated. In our numerical evaluation we used the very general approach which just take into consideration the truncated value of $u(t)$.

4. Numerical solution by the Runge-Kutta of order 4

The solution is computed using the Matlab software. The second order differential équation is transformed into a first order system and then solved by the Runge-Kutta of order 4. This solution has been obtained by the Runge-Kutta of order 4. The solutions given by the two methods start at the same point and are very similar in function of the order of the truncated function in the (DTM) see the graph below:

5. Numerical results by the reduced differential transform method

We consider the truncated formula of the RDTM of order 4 see Fig. 1 below

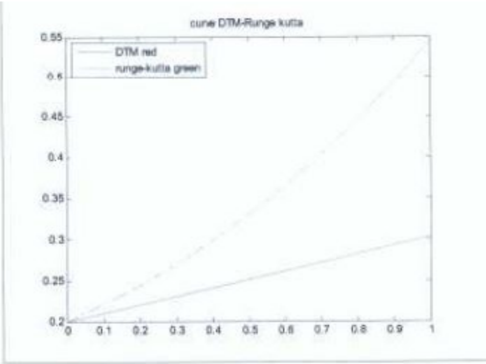

Figure 1

We have truncated the DTM at order 4. The calculations are very huge although we use Matlab truncated of order n arbitrary can be done and the curves must be sufficiently close.

6. Evaluation Procedure for the Coefficients of the Transformed Function

To evaluate the coefficients of our solution deduced by the differential transform method, we use the equation $\frac{d^2u}{dt^2} + F(u) = 0$ with the initial values

$$u(0) = \alpha \quad (15)$$
$$u'(0) = \beta \quad (16)$$

We apply the transform operator to the members of equations (15) using the linearity of the operator and the property P1 we have for any $k \in N$, the relation:

$$\frac{(k = 2)!}{k!} U(k+2) + M_k F(U) = 0$$

Since u is unknown, the use of the relation shall be done carefully. Using a truncation of the solution at order N i.e. we take $u(x) = \sum_{k=0}^{N} a_k x^k$; recalling also that the $u(k) = \frac{1}{k!}\left[\frac{d^k u}{dt^k}\right]_{t=t_0}$ we can deduce the following relation

$$2U(2) + F\big(U(0)\big) = 0 \quad (i)$$
$$6U(3) + M_1 F(u) = 0 \quad (ii)$$

In the second equation U(0), U(1), U(2) are known but M_1F can be obtained as U(k) $k \geq 3$in a nonlinear way.

Let us get $g(t) = F\big(u(t)\big) = \sum_{k=O}^{N} a_k t^k$by Taylor expansion about 0 the $a_k = a_k(a_0, a_1, \dots, a_N); k = 0,1,\dots,N$

We can then evaluate successively he coefficients U(k) and solving the equations in the form (i) and (ii). For n = 4 we obtain the following relations.

$$g(t) = \frac{a_1}{(1+a_0^2)} + (a_1 - 2 * a_0^2/(1+a_0^2) * a_1)/(1+a_0^2)x + (a_2 - a_1/(1+a_0^2) * (2 * a_0 * a_2 * a_0^2 + 2 \cdot a))$$

The notations are those of MATLAB or OCTAVE.
We can then deduce the different values. We observe that the first coefficient depend only on $a_0 = F(U\,(0))$, the second on a_0 and a_1 and so on.
We also use the remark that:
The coefficient of x^k in the right hand side is

$$\frac{1}{k!}\frac{d^k F}{dt^k}\big(u(0)\big)$$

Let us evaluate the coefficient for our example with $F(u) = \frac{u}{1+u^2}$ the solution for any F analytic can be carried out, $u(0) = h, u'(0) = 0$

Remark
In a forthcoming paper other cases are going to be treated.

Conclusion1
In this paper we have proved that the Differential Transform Method can be used successfully to approximate with very high accuracy nonlinear ordinary differential equations. We have indicated some technical procedures that can be used with the symbolic software for our computation. The procedure is very general to be used in a very high class of examples and application. Many physical models being in this form.

II. Adomian Decomposition Method in Solving a Class of Nonlinear Differential Equation

2.1 The decomposition method is one of the new numerical method developed to solve with arbitrary higher order of accuracy. The Adomain method has been introduced and developed by Adomain. It is very efficient for obtaining closed form and numerical approximation of a large class of ordinary, partial and even algebraic equations arising from problem in natural sciences and engineering.

We consider the following class of ordinary differential equations of second order

$$\frac{d^2u}{dt^2} + F(u) = 0 \quad (1)$$
$$u(O) = \alpha; u'(0) = \beta \quad (2)$$

The problem (1) and (2) has been considered recently by (Tchoua and Ita 2012) where they gave a numerical solution of this class of problem by the differential reduced method.

In this paper the Adomain is used to solve the same problem and the results compared. Let us present the general structure of the Adomain method.

2.2 The Adomain Decomposition Method

Our equation can be written in the form

$$Lu - N(u) = f \quad (3)$$

where L is a linear operator and N a nonlinear operator. The linear operator is assumed invertible. The Adomain method consist of approximating the solutions of (1) - (2) as an infinite series

$$u = \sum_{n=0}^{\infty} u_n \quad (4)$$

and expressing the nonlinear operator in the form

$$N(y) = \sum_{n=0}^{\infty} A_n$$

When A_n are the adomain polynomials of $y_0, y_1 \dots y_n \dots$ as defined by (Adomain 1994) where

$$A_n = \frac{1}{n!}\frac{d}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right] \quad i = 0,1,2 \dots \quad (5)$$

Applying the inverse of the linear operator L^{-1} to the members of after substituting (4) and (5) we obtain the following relations

$$\sum_{n=0}^{\infty} u_n - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) = L^{-1}(f) \quad (6)$$

We can deduce the following iterative relations:

$$u_0 = L^{-1}f$$
$$u_{n+1} = L^{-1}((A_n)) \quad (7)$$

We apply to the case

2.3 Application

Example 1

$$N(u) = \frac{u}{1+u^2}$$
$$A_0 = N(U_0)$$
$$A_1 = U_1 N(U_0)$$
$$A_2 = U_2 N(U_0) + \frac{u_1^2}{2} N''(U_0)$$
$$A_3 = U_3 N(U_0) + U_1 U_2 N''^{(U_0)\frac{u_1^2}{6}} N'''^{(U_0)} \quad (8)$$

The coefficients A_n can be estimated recursively. The general formula can be deduced from the formula established by Abbaoul and Cherruault (1994).

$$A_n = \sum_{k=1} N^{(k)}(y_0)\frac{\left[\sum_{p_1+p_2\dots p_3} y_{p1}, y_{p2} \dots y_{pn}\right]}{p_1!\, p_2! \dots p_n!}, n \geq 1$$

We can then deduce the value of

$$y_n = \int_0^t \int_0^r A_n dr dt$$

We choose in our investigation the case p = 0 in that case and

$$-N(u) = \frac{u}{1+u^2}$$

Solution

$$N'^{(u)} = \frac{1}{1+u^2} - 2\frac{u^2}{(1+u^2)^2};$$

$$\frac{d^2}{du^2}\left[\frac{u}{1+u^2}\right] = \left[8\frac{u^3}{(1+u^2)^2} - 6\frac{u}{(1+u^2)^2}\right] = N'^{(u)}(10)$$

$$\frac{d^3}{du^2}\left[\frac{u}{1+u^2}\right] = \left[48\frac{u^2}{(1+u^2)^3} - 48\frac{u^4}{(1+u^2)^4} - \frac{6}{(1+u^2)^2}\right] = N^{(3)}(u)$$

We can evaluate the derivative of all orders using the symbolic computing software MATLAB and SCIENTIFIC WORKPLACE

$$y_0 = h$$

$$y_1 = h\frac{t^2}{2} = b_1 t^2$$

$$A_i = y_i N(y_0) = \frac{ht^2}{2}\left[\frac{1-h^2}{(1+h^2)^2}\right]$$

$$y_2(t) = \int_0^t \int_0^\tau A_1 d\tau\, ds = \frac{h^2}{2}\left[\frac{1-h^2}{(1+h^2)^2}\right]\frac{t^2}{12} = \frac{h^2}{24}\left[\frac{1-h^2}{(1+h^2)^2}\right]t^4 = b_2 t^4$$

$$A_1 = y_2 N'(y_0) + \frac{y_1^2}{2} N''(y_0) = y_2(t)\left[\frac{1-h^2}{(1+h^2)^2}\right] + \frac{y_1(t)\cdot y_1(t)}{2}\left[8\frac{h^3}{(1+h^2)^3} - 6\frac{h}{(1+h^2)^2}\right]$$

We can then deduce the value of y_3

$$y_3 = \int_0^t \int_0^\tau A_2(s) ds\, d\tau = \frac{1}{30} b_2 t^6 + \frac{1}{30} b_1^2 t^6$$

We can then evaluate the solution third order truncation

$$u(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t)$$

Example 2

Consider the equation

$$\frac{du}{dt} + g(t,u) = 0$$
$$u(0) = h;$$

where

$$g(t,u) = \int_0^t h^p(s)\, ds; p > 0$$

And arbitrary positive real. This equation is equivalent to the second order differential equation

$$\frac{d^2u}{dt^2} + u^p = 0$$
$$u(0) = h; u'(0) = 0$$

Solution by Adomian Decomposition Method

The equation can be put in the form

$$Lu - N(u) = 0$$

L is the linear part and N(u) is the nonlinear part

$$N(u) = -u^p$$
$$N(u) = \sum_{i=0}^{\infty} A_i$$

the A_i being evaluated according to the equation (9)

$$u(t) = y_0(t) + \sum_{i=0}^{\infty} y_0(t)$$

$$-A_0 = y_0^p$$

$$-A_1 = p y_0^{p-1} y_1$$

$$-A_2 = \frac{p(p-1)}{2} y_0^{p-2} y_1^2 + p y_0^{p-1} y_2$$

$$-A_3 = \frac{p(p-1)(p-2)}{6} y_0^{p-3} y_1^3 + p(p-1) y_0^{p-2} y_1 y_2 + p y_0^{p-1} y_3$$

The values of y_1 given by the formula (10). We have the following:

$$y_0(t) = h$$

$$y_1(t) = \frac{h}{2} t^2$$

$$y_2(t) = \frac{p h^{p-1} t^4}{12}$$

We can get more terms using the equation (9).

The solution has been obtained in Tchoua and Ita (2012) for the example 1, for the differential transform method.

III. Conclusion

We have investigated some numerical methods of solving ODE of same class which yield analytical solutions for arbitrary order.

We have solved a class of differential equation by the Differential Transform Method and the Adomain Decomposition Method. It appears that these methods are very accurate for small values of the variable but for large values we shall adapt the method by changing the point of evaluation of the coefficients. They also appear as very powerful in solving nonlinear ordinary differential equations

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