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Some Algebra of Leibniz Rule for Fractional Calculus

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***Abstract* – It is well known that the Leibniz rule has a binomial representation of derivative for the product of two functions. This has an identical form which is the power of sum of two functions. Since the power of sum can be fraction, we can extend the derivative to the fractional power, i.e., Riemann-Liouville derivative and Caputo derivative. And then, the generalized Leibniz rule is derived in which the form is the fractional power of the binomial representation. Moreover, we describe the Riemann-Liouville fractional derivative and Caputo derivative of production of two functions as the sum of integer powers. On the other hand, we introduce the projection operator and calculate some algebra.**

***Keywords* – Cauchy Formula, Riemann-Liouville Derivative, Caputo Derivative, Generalized Leibniz Rule, Quantization, Projection Operator.**

I. Introduction

As an originator of differentiation and integration, G.W. Leibniz (1646-1716) published a writing pertaining to the differentiation of a product of some functions in 1710 [1]. He described the similarity between the power of the sum and the derivative of the product. That is the resemblance of the coefficients between $p^e(x+y)$ and $d^e(xy)$. He derived the power of sum as follows.

$$p^e(x+y) = 1x^e + \frac{e}{1}x^{e-1}y + \frac{e(e-1)}{1\cdot 2}x^{e-2}y^2 + \frac{e(e-1)(e-2)}{1\cdot2\cdot3}x^{e-3}y^3 \& c\,.$$

And then, Leibniz replaced $p^e(x+y)$ with $d^e(xy)$. He used the notation d as infinitesimal variation and started from $e=1$.

$$d(xy) = (x+dx)(y+dy) - xy = ydx + xdy \quad ;\, dxdy \ll ydx,\ xdy.$$

Then, for any nonnegative integer e, the differentiation of product can be expressed in a binomial form. In general, Leibniz demonstrated the rule, called Leibniz Rule as follow.

$$d^e(xy) = 1d^e x d^0 y + \frac{e}{1}d^{e-1}xd^1y + \frac{e(e-1)}{1\cdot 2}d^{e-2}xd^2y + \frac{e(e-1)(e-2)}{1\cdot2\cdot3}d^{e-3}xd^3y \& c.$$

Above equations show that $p^e(x+y)$ and $d^e(xy)$ have an identical binomial form. Aside from the Leibniz Rule, he also took note of the fractional calculus; it is the extended version of differentiation and integration to the real or complex power. A question was raised in year 1695 in the letter from M. L'Hôpital (1661-1704) to G. W. Leibniz, which sought the meaning of Leibniz's notation $\frac{d^ny}{dx^n}$ for the derivative of order $n \in \mathbb{N}_0 = \{0,1,2,\cdots\}$ when $n=\frac{1}{2}$? In his reply dated 30 September 1695, Leibniz wrote to L'Hôpital as follows: "$\cdots$ This is an apparent paradox from which, on the day, useful consequences will be drawn. $\cdots$" [5]. Thereafter, mathematicians have made efforts on the concept, and the fractional calculus have become a prosperous area of mathematics and physics for the last few decades.

II. Leibniz Rule

The theorem of differentiation and integration should be drawn from the concept of limit. However, there was not so definite concept of limit when he published the paper, so the Leibniz rule needs to be explained in modern form.

Theorem 1 (Leibniz rule) For n times differentiable functions f and g on $[a,b]$,

$$(fg)^{(n)} = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}g^{(n-k)} \qquad \text{on}(a,b).$$

Proof. Omitted. We can easily prove it by mathematical induction with combinatorics.

Hence, it suffices for any nonnegative integer n. This form is identical to the binomial representation for the power of sum such that

$$(f+g)^n = \sum_{k=0}^{n}\binom{n}{k} f^k g^{n-k}.$$

Definition 1 (integral and differential operator) For simplicity, we define integral operator and differential operator as follows.

$${}_aJ_x[t] = \int_a^x dt$$
$$D[t] = \frac{\partial}{\partial t}$$

Here, $[t]$ means the domain of operators. We can extend the definition to iterated forms as

$${}_aJ_x^n[\vec{x}] = {}_aJ_x[x_1]\,{}_aJ_{x_1}[x_2]\cdots{}_aJ_{x_{n-1}}[x_n] = \int_a^x dx_1\int_a^{x_1}dx_2\cdots\int_a^{x_{n-1}}dx_n$$

and

$$D^n[\vec{x}] = D[x_1]D[x_2]\cdots D[x_n] = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n},$$

for $n\in\mathbb{N}_0$ and $\vec{x} = (x_1,x_2,\cdots,x_n)$.

Thus, for an analytic function f on $[a,b]$, we can extend the integration and differentiation to the n-th power.

$${}_aJ_x^n[\vec{x}]f(x_n) = \int_a^x dx_1\int_a^{x_1}dx_2\cdots\int_a^{x_{n-1}}dx_n\, f(x_n),$$

and

$$D^n[\vec{x}]f(x_n) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_n}\, f(x_n).$$

For the case of $\vec{x}=(t,t,\cdots,t)$, we describe the above operators ${}_aJ_x^n[t]$ and $D^n[t]$ rather than ${}_aJ_x^n[\vec{x}]$ and $D^n[\vec{x}]$. Since we can treat the integration as an inverse of the differentiation, we can describe the integration and differentiation as a unified form as follows.

$$ {}_aD_x^n[\vec{x}] = \begin{cases} D^n[\vec{x}] & ,n \geq 0 \\ {}_aJ_x^n[\vec{x}] & ,n < 0 \end{cases} $$

III. Fractional Calculus

As we mentioned before, what about the fractional power of differentiation and integration? Cauchy formula can be the clue to this question.

Theorem 2 (Cauchy formula) [6] For an integrable function functions f on $[a,b]$, the n times integration of f is reduced to a single integration as

$$ {}_aJ_x^n[\vec{x}]f(x_n) = \int_a^x dx_1 \int_a^{x_1} dx_2 \cdots \int_a^{x_{n-1}} dx_n\, f(x_n) $$
$$ = \frac{1}{\Gamma(n)} \int_a^x dx_n (x - x_n)^{n-1} f(x_n)\,. $$

Proof. Omitted. We can easily prove it by mathematical induction.

Hence, we re-define the power of integral and differential operator as one variable.The Cauchy formula provides us the idea of extension for the power of integral from non-negative integer to real number. If the power $n \in \mathbb{N}_0$ is changed to $\alpha \in \mathbb{R}$, the operator is named Riemann-Liouville fractional integral operator.

Definition 2 (Riemann-Liouville fractional integral operator)[7, 11] The Riemann-Liouville fractional integral operator ${}_aJ_x^\alpha[\vec{x}]$ is defined by

$$ {}_aJ_x^\alpha[t] = \frac{1}{\Gamma(\alpha)} \int_a^x dt (x-t)^{\alpha-1} \quad \text{for } \alpha > 0. $$

Since the gamma function is continuous for positive numbers, $\Gamma(\alpha)$ is well-defined for $\alpha > 0$. Therefore, ${}_aJ_x^\alpha[t]$ is continuous for $\alpha > 0$. Note that the integration is a convolution form with a kernel function $\Phi_\alpha(x-t) := (x-t)^{\alpha-1}/\Gamma(\alpha)$ of a power-law type. That is

$$ {}_aJ_x^\alpha[t]f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x dt (x-t)^{\alpha-1} f(t) $$
$$ = \int_a^x dt \Phi_\alpha(x-t) f(t) $$
$$ = (\Phi_\alpha * f)(x). $$

Then, the Laplace transformation is

$$ L\{ {}_aJ_x^\alpha[t]f(t) \} = L\{(\Phi_\alpha * f)(x)\} = L\{\Phi_\alpha(x)\}L\{f(x)\}. $$

This means that the Laplace transformation of fractional integration is a simple multiplication of each transformation [6]. Note that the Riemann-Liouville fractional integral operator satisfies the semi-group property, the non-negative numbers α and β,

$$ {}_aJ_x^\alpha[x']\, {}_aJ_{x'}^\beta[t] = {}_aJ_x^{\alpha+\beta}[t]. $$

By symmetry, we may design the fractional differential operator. Unfortunately, the definition of differential operator is non-trivial. However, there are two representative definitions of the fractional differential operators based on Riemann-Liouville fractional integral operator.

Definition 3 (Riemann-Liouville fractional differential operator) [7, 11] Riemann-Liouville fractional differential operator ${}^{RL}_aD_x^\alpha[t]$ is defined by

$$ {}^{RL}_aD_x^\alpha[t] = D^n[x]\, {}_aJ_x^{n-\alpha}[t] \qquad \text{for } 0 \leq n-\alpha < 1. $$

That is

$$ {}^{RL}_aD_x^\alpha[t] = \frac{\partial^n}{\partial x^n} \frac{1}{\Gamma(n-\alpha)} \int_a^x dt (x-t)^{n-\alpha-1} $$
$$ \text{for } 0 \leq n-\alpha < 1. $$

Definition 4 (Caputo fractional differential operator) [10] Caputo fractional differential operator ${}^C_aD_x^\alpha[t]$ is defined by

$$ {}^C_aD_x^\alpha[t] = {}_aJ_x^{n-\alpha}[t] D^n[t] $$
$$ \text{for } 0 \leq n-\alpha < 1. $$

That is

$$ {}^C_aD_x^\alpha[t] = \frac{1}{\Gamma(n-\alpha)} \int_a^x dt (x-t)^{n-\alpha-1} \frac{\partial^n}{\partial t^n} $$
$$ \text{for } 0 \leq n-\alpha < 1. $$

As the fractional number contains integer, the fractional differentiation may approach the original differentiation and integration where α get closer to an integer. The necessity condition is that the fractional differentiation satisfies the *fundamental theorem of fractional calculus* (*FTFC*) and *Newton-Leibniz formula*. The Caputo fractional derivative satisfies *FTFC* for every non-negative number α, and *Newton-Leibniz formula* for $0 < \alpha < 1$.

That is

$$ {}^C_aD_x^\alpha[x']\, {}_aJ_{x'}^\alpha[t]f(t) = {}_aJ_x^{n-\alpha}[x']D^n[x']\, {}_aJ_{x'}^\alpha[t]f(t) $$
$$ = {}_aJ_x^{n-\alpha}[x']D^n[x']\, {}_aJ_{x'}^n[x'] J_{x'}^{\alpha-n}[t]f(t) $$
$$ = {}_aJ_x^{n-\alpha}[x'] J_{x'}^{\alpha-n}[t]f(t) $$
$$ = f(x) $$
$$ \text{for } \alpha > 0, $$

and

$$ {}_aJ_x^\alpha[x']\, {}^C_aD_{x'}^\alpha[t]f(t) = {}_aJ_x^\alpha[x']\, {}_aJ_{x'}^{1-\alpha}[t]D[t]f(t) $$
$$ = {}_aJ_x\,[t]D[t]f(t) = f(x) - f(a) $$
$$ \text{for } 0 < \alpha < 1. $$

However, Riemann-Liouville fractional derivative satisfies the *FTFC* almost everywhere for $\alpha > 0$, and does not satisfy *Newton-Leibniz formula*[9]. Above all, we cannot find the physical meaning for Riemann-Liouville operator for a constant: Riemann-Liouville fractional derivative of a constant is not zero. Thus, the Caputo fractional differential operator is more useful. Recent monographs and symposia proceedings have highlighted the application of fractional calculus in physics, continuum mechanics, signal processing, and electromagnetism [8].

IV. Generalized Leibniz Rule

As the symmetry is discovered between non-negative power of sum and derivative of product, it is only natural that one may try to discover the symmetry between real power of sum and fractional derivative of product. Before extending the power of derivative for the generalized Leibniz rule, the following lemmas should be considered.

Lemma 1 Let $f(x)$ be an analytic function on[a, b], then the Riemann-Liouville integration operator satisfies that

$$\left({}_aJ_x^\beta f\right)(x) = \sum_{m=0}^{\infty}\binom{-\beta}{m}\frac{f^{(m)}(x)}{\Gamma(m+\beta+1)}(x-a)^{m+\beta}$$

for $\beta \geq 0$.

Proof. The equation is satisfied trivially for $\beta = 0$. Now for $\beta > 0$,

$$\left({}_aJ_x^\beta f\right)(x) = \frac{1}{\Gamma(\beta)}\int_a^x dt(x-t)^{\beta-1}f(t)$$

$$= \frac{1}{\Gamma(\beta)}\int_a^x dt(x-t)^{\beta-1}\sum_{m=0}^{\infty}\frac{(-1)^m f^{(m)}(x)}{\Gamma(m+1)}(x-t)^m$$

; Taylor series expansion of $f(t)$ around x.

$$= \sum_{m=0}^{\infty}\frac{(-1)^m f^{(m)}(x)}{\Gamma(\beta)\Gamma(m+1)}\int_a^x dt(x-t)^{m+\beta-1}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m f^{(m)}(x)}{\Gamma(\beta)\Gamma(m+1)(m+\beta)}(x-a)^{m+\beta}$$

$$= \sum_{m=0}^{\infty}\binom{-\beta}{m}\frac{f^{(m)}(x)}{\Gamma(m+\beta+1)}(x-a)^{m+\beta}.$$

Lemma 2[13] Let $f(x)$ be an analytic function on $[a, b]$, then the Riemann-Liouville operator satisfies that

$$\left({}_a^{RL}D_x^\alpha f\right)(x) = \sum_{m=0}^{\infty}\binom{\alpha}{m}\frac{f^{(m)}(x)}{\Gamma(m-\alpha+1)}(x-a)^{m-\alpha}$$

for $\alpha \in \mathbb{R}$.

Proof. For $\alpha \leq 0$, the equation is satisfied by setting $\beta = -\alpha$ on Lemma 1. Now suppose that $0 \leq n-\alpha < 1$ for $n \in \mathbb{N}$. Then, by definition of Riemann-Liouville derivative,

$$\left({}_a^{RL}D_x^\alpha f\right)(x) = \left(D^n\, {}_aJ_x^{n-\alpha} f\right)(x)$$

$$= D^n\left[\sum_{m=0}^{\infty}\binom{\alpha-n}{m}\frac{f^{(m)}(x)(x-a)^{m+n-\alpha}}{\Gamma(m+n-\alpha+1)}\right]$$

; Lemma 1

$$= \sum_{m=0}^{\infty}\binom{\alpha-n}{m}\frac{1}{\Gamma(m+n-\alpha+1)}$$

$$\times\sum_{k=0}^{n}\binom{n}{k}f^{(m+k)}(x)\frac{\Gamma(m+n-\alpha+1)}{\Gamma(m+k-\alpha+1)}(x-a)^{m+k-\alpha}$$

; Classical Leibniz Rule

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{n}\binom{\alpha-n}{m}\binom{n}{k}\frac{f^{(m+k)}(x)(x-a)^{m+k-\alpha}}{\Gamma(m+k-\alpha+1)}$$

$$= \sum_{m+k=0}^{\infty}\binom{\alpha}{m+k}\frac{f^{(m+k)}(x)(x-a)^{m+k-\alpha}}{\Gamma(m+k-\alpha+1)}$$

$$= \sum_{m'=0}^{\infty}\binom{\alpha}{m'}\frac{f^{(m')}(x)(x-a)^{m'-\alpha}}{\Gamma(m'-\alpha+1)}.$$

Consequently, the generalized Leibniz rule is induced for the fractional power of derivative.

Theorem 3 (Generalized Leibniz rule for Riemann-Liouville derivative) [7, 12-13] For analytic functions f and g on $[a, b]$, the Leibniz rule holds for $\alpha \in \mathbb{R}$. That is

$${}_a^{RL}D_x^\alpha(fg) = \sum_{k=0}^{\infty}\binom{\alpha}{k}D^k(f)\,{}_a^{RL}D_x^{\alpha-k}(g)$$

Proof. Since f and g are analytic, fg is also analytic. Thus,

$${}_a^{RL}D_x^\alpha(fg) = \sum_{m=0}^{\infty}\binom{\alpha}{m}\frac{(fg)^{(m)}}{\Gamma(m-\alpha+1)}(x-a)^{m-\alpha}$$

; Lemma 2

$$= \sum_{m=0}^{\infty}\binom{\alpha}{m}\frac{(x-a)^{m-\alpha}}{\Gamma(m-\alpha+1)}\sum_{k=0}^{m}\binom{m}{k}f^{(k)}g^{(m-k)}$$

; Classical Leibniz Rule

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{\alpha}{m}\binom{m}{k}\frac{(x-a)^{m-\alpha}}{\Gamma(m-\alpha+1)}f^{(k)}g^{(m-k)}$$

$$= \sum_{k=0}^{\infty}\sum_{m=k}^{\infty}\binom{\alpha}{k}\binom{\alpha-k}{m-k}\frac{(x-a)^{m-\alpha}}{\Gamma(m-\alpha+1)}f^{(k)}g^{(m-k)}$$

$$;\binom{\alpha}{m}\binom{m}{k} = \binom{\alpha}{k}\binom{\alpha-k}{m-k}$$

$$= \sum_{k=0}^{\infty}\binom{\alpha}{k}f^{(k)}\sum_{m=k}^{\infty}\binom{\alpha-k}{m-k}$$

$$\times\frac{g^{(m-k)}}{\Gamma((m-k)-(\alpha-k)+1)}(x-a)^{(m-k)-(\alpha-k)}$$

$$= \sum_{k=0}^{\infty}\binom{\alpha}{k}f^{(k)}\,{}_a^{RL}D_x^{\alpha-k}(g).$$

; Lemma 2

As in the classical Leibniz Rule, the form of generalized Leibniz Rule for Riemann-Liouville derivative has the identical binomial form compared to fractional power of sum such that

$$P^k(f+g) = \sum_{k=0}^{\infty}\binom{\alpha}{k}P^k(f)P^{\alpha-k}(g).$$

Unfortunately, there is no symmetric representation for the Caputo differential operator. However, we can use the relation between Caputo and Riemann-Liouville operator to represent the Caputo operator as

V. Some Algebra of Fractional Integral and Differential Operators

Now, we will search for methods of quantizing the power of fractional operators. First, we can quantize the fractional integral operator as follows.

Lemma 3 The Riemann-Liouville fractional integral operator ${}_aJ_x^\beta$ can be described as the sum of integer powers. That is

$${}_aJ_x^\beta = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{\beta}{k}\binom{k}{j}(-1)^{k-j}\,{}_aJ_x^j.$$

Proof. ${}_aJ_x^\beta = \left[\left({}_aJ_x - I\right) + I\right]^\beta$

$$= \sum_{k=0}^{\infty}\binom{\beta}{k}\left({}_aJ_x - I\right)^k I^{\beta-k}$$

$$= \sum_{k=0}^{\infty}\binom{\beta}{k}\sum_{j=0}^{k}\binom{k}{j}\,{}_aJ_x^j(-I)^{k-j}$$

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$= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{\beta}{k}\binom{k}{j}(-1)^{k-j} \ {}_aJ_x^j.$

Note that we cannot change the order of sums, unless the sum of integer power causes a contradiction as follow.

$$
\begin{aligned}
{}_aJ_x^\beta &= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{\beta}{k}\binom{k}{j}(-1)^{k-j}\ {}_aJ_x^j \\
&= \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{\beta}{k}\binom{k}{j}(-1)^{k-j}\ {}_aJ_x^j \\
&= \sum_{j=0}^{\infty}\sum_{k-j=0}^{\infty}\binom{\beta}{j}\binom{\beta-j}{k-j}1^{\beta-k}(-1)^{k-j}\ {}_aJ_x^j \\
&= \sum_{j=0}^{\infty}\binom{\beta}{j}(1-1)^{\beta-j}\ {}_aJ_x^j \\
&= 0.\text{(contradiction)}
\end{aligned}
$$

Now we can describe the Riemann-Liouville fractional derivative and Caputo fraction derivative as the sum of integer powers.

Theorem 4 (quantization of generalized Leibniz rule) The Riemann-Liouville fractional derivative of the product of two functions, i.e., generalized Leibniz rule can be describe as the sum of integer powers. That is

$$
{}_a^{RL}D_x^\alpha(fg) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{n-j}\binom{n-\alpha}{k}\binom{k}{j}\binom{n-j}{i} \times (-1)^{k-j}D^i(f)\ {}_aD_x^{n-j-i}(g).
$$

Proof.
$$
\begin{aligned}
{}_a^{RL}D_x^\alpha(fg) &= D^n\ {}_aJ_x^{n-\alpha}(fg) \\
&= D^n\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{n-\alpha}{k}\binom{k}{j}(-1)^{k-j}\ {}_aJ_x^j(fg) \quad \text{; Lemma 3} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{n-\alpha}{k}\binom{k}{j}(-1)^{k-j}\ {}_aD_x^{n-j}(fg) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{i=0}^{n-j}\binom{n-\alpha}{k}\binom{k}{j}\binom{n-j}{i} \times (-1)^{k-j}D^i(f)\ {}_aD_x^{n-j-i}(g).
\end{aligned}
$$

We can also quantize the Caputo derivative of functions using quantization of the fractional integral operator.

Theorem 5 (quantization of Caputo fractional derivative) The Caputo fractional derivative of the product of two functions can be describe as the sum of integer powers. That is

$$
{}_a^{C}D_x^\alpha(fg) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{i=0}^{\infty}\binom{n-\alpha}{k}\binom{k}{j}\binom{n}{l}\binom{-j}{i} \times (-1)^{k-j}D^{l+i}(f)\ {}_aD_x^{n-l-j-i}(g)\ .
$$

Proof.
$$
\begin{aligned}
{}_a^{C}D_x^\alpha(fg) &= {}_aJ_x^{n-\alpha}D^n(fg) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{n-\alpha}{k}\binom{k}{j}(-1)^{k-j}\ {}_aJ_x^j D^n(fg) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{n-\alpha}{k}\binom{k}{j}\binom{n}{l} \times (-1)^{k-j}\ {}_aJ_x^j D^l(f)D^{n-l}(g) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{n}\sum_{i=0}^{\infty}\binom{n-\alpha}{k}\binom{k}{j}\binom{n}{l}\binom{-j}{i} \times (-1)^{k-j}D^{l+i}(f)\ {}_aD_x^{n-l-j-i}(g)
\end{aligned}
$$

Now, we may consider the direct calculation of operators. Note that the *fundamental theorem of calculus* (*FTC*) and *Newton-Leibniz formula* are expressed as follows.

$$
\begin{aligned}
D[x']\ {}_aJ_{x'}[t]f(t) &= f(x) \\
{}_aJ_x\ [t]D[t]f(t) &= f(x) - f(a).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(D\ {}_aJ_x\ )f &= If = f \\
(\ {}_aJ_x\ D)f &= f(x) - f(a)
\end{aligned}
$$

for simplicity. Hence,

$$
{}_aJ_x\ D = D\ {}_aJ_x\ + [\ {}_aJ_x\ , D] = I + [\ {}_aJ_x\ , D]
$$

where the parenthesis means a commutator.

Proposition 1 (projection operator) ${}_aJ_x\ D$ is a projection operator

Proof.
$$
\begin{aligned}
(\ {}_aJ_x\ D)^2 &= (\ {}_aJ_x\ D)(\ {}_aJ_x\ D) = \ {}_aJ_x\ (D\ {}_aJ_x\ )D \\
&= \ {}_aJ_x\ (D\ {}_aJ_x\ )D = \ {}_aJ_x\ ID = \ {}_aJ_x\ D.
\end{aligned}
$$

If an operator is equal to its square, that is a projection operator. Thus, ${}_aJ_x\ D$ is a projection operator.

Proposition 2 $\exp(\ {}_aJ_x\ ) - I$ is in the space projected by the projection operator ${}_aJ_x\ D$.

Proof.
$$
\begin{aligned}
{}_aJ_x\ D\exp(\ {}_aJ_x\ ) &= \ {}_aJ_x\ D\left(I + \ {}_aJ_x\ + \frac{{}_aJ_x^2}{2!} + \frac{{}_aJ_x^3}{3!} + \cdots\right) \\
&= \ {}_aJ_x\ D + \ {}_aJ_x\ + \frac{{}_aJ_x^2}{2!} + \frac{{}_aJ_x^3}{3!} + \cdots .
\end{aligned}
$$

Therefore,

$$
{}_aJ_x\ D[\exp(\ {}_aJ_x\ ) - I] = \exp(\ {}_aJ_x\ ) - I
$$

That is, the projection of $\exp(\ {}_aJ_x\ ) - I$ is $\exp(\ {}_aJ_x\ ) - I$ itself.

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