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A Study on Order Statistics from Nonidentically Distributed Kumaraswamy Variables

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Abstract – **In this work I deduced the moments of order statistics for independent nonidentically (inid) distributed Kumaraswamy random variables. Probability density function and cumulative distribution function for the r[th] inid order statistics at r = 1, n = 3 and their graphs are found. Numerical calculations of moments for different values of the parameters from inid order statistics of this distribution are given. The first two moments and the variance of the second order statistics of sample size n = 3 are computed. Moments of the last order statistics of sample size n = 2 in the presence of multiple outliers are obtained. All calculations are tabulated.**

Keywords – **Order Statistics, Moments, Non-identically Distributed, Kumaraswamy Distribution.**

I. Introduction

There are three trends have emerged in the literature to obtain moments of order statistics (OS) for (inid) distributed random variables. One of these trends is initiated by (Balakrishnan, 1994). He exploited the known relation between the probability density function (pdf) and the cumulative distribution function (cdf). Other trend is applied to distributions with cdf in the form: $F(x) = 1 - \varphi(x)$, this trend was established by Barakat and Abdelkader (2003). Last trend was established by Jamjoom and Alsaiary (2011). In this last trend researchers using the moment generating function technique. In this paper we computed the moments of OS arising from inid distributed Kumaraswamy random variables (rvs) using second trend.

The study of OS from inid rvs is started in the literature by (Vaughan and Venables, 1972; David, 1981; Bapat and Beg, 1989; David and Nagaraja, 2003). (Balakrishnan, 1994) derived recurrence relations for single and product moments of OS from INID rvs for the Exponential distribution. Childs and Balakrishnan (2006) derived the moments of OS from INID rvs for Logistic distribution. (Mohamed *et al.*, 2007) else applied Balakrishnan method for doubly truncated Lomax , Weibull-Gamma, Burr XII and for general class of distributions. The second trend was applied by (Barakat and Abdelkader, 2000) in weibull distribution and then they generalized this method in (2003) and used it in Erlang distribution. The survival function of Gamma and Beta distributions used by Abdelkader (2004) and (Abdelkader, 2008) to compute the moments of OS from inid rvs for beta distribution. AL-Saiary (2015) applied the third trend to Standard type II Generalized logistic rvs.

II. Kumaraswamy Distribution

The probability density function of a random variable X from Kumaraswamy distribution is given by

$$f(x) = \begin{cases} a\,b\,x^{a-1}\left[1-x^{a}\right]^{b-1}, a>0,\ b>0,\ 0\le x\le 1 \\ 0,\ \text{otherwise} \end{cases} \qquad (1)$$

The cumulative distribution function is given as

$$F(x) = 1-\left[1-x^{a}\right]^{b}, a>0,\ b>0,\ 0\le x\le 1 \qquad (2)$$

a and b are shape parameters. See Kumaraswamy (1980).

IV. Nonidentical Order Statistics from Kumaraswamy Distribution

Let $X_1, X_2, \ldots, X_n$ be independent random variables having cumulative distribution functions $F_1(x), F_2(x), \ldots, F_n(x)$ and probability density functions $f_1(x), f_2(x), \ldots, f_n(x)$, respectively. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ denote the order statistics obtained by arranging the $n\ X_i\,s$ in increasing order of magnitude. Then the pdf and the cdf of the r[th] order statistic $X_{r:n}$ $(1 \le r \le n \le)$ can be written as:

$$f_{r:n}(x) = \frac{1}{n!\,\beta(r, n-r+1)} \times \sum_{p} \prod_{a=1}^{r-1} F_{i_a}(x)\, f_{i_r}(x) \prod_{c=r+1}^{n} \left\{1-F_{i_c}(x)\right\} \qquad (3)$$

Where $\sum_{p}$ denotes the summation over all $n!$ permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots n)$. (Bapat and Beg, 1989) put f (x) and F (x) of the r[th] OS from inid rvs in the form of permanent as:

$$f_{r:n}(x) = \frac{1}{n!\,\beta(r, n-r+1)} \times per\left[\underbrace{F(x)}_{r-1}\ \underbrace{f(x)}_{1}\ \underbrace{\{1-F(x)\}}_{n-r}\right] \qquad (4)$$

$$F_{(r)}(x) = \sum_{j=r}^{n} \sum_{p_j} \prod_{a=1}^{j} F_{i_a}(x) \prod_{a=j+1}^{n} \left[1-F_{i_a}(x)\right] \qquad (5)$$

$\sum_{p_j}$ is all permutations of $(i_1, i_2, \cdots, i_n)$ for $(1, \cdots, n)$ which satisfy $i_1 < i_2 < \cdots < i_j$ and $i_{j+1} < i_{j+2} < \cdots < i_n$. And using permanent as

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$$F_{(r)}(x) = \sum_{i=r}^{n} \frac{1}{(n+1)!\,\beta(i+1, n-i+1)} \times per \begin{bmatrix} F_1(x) & 1-F_1(x) \\ \vdots & \\ \underbrace{F_n(x)}_{i} & \underbrace{1-F_n(x)}_{n-i} \end{bmatrix}, -\infty < x < \infty \quad (6)$$

we can obtain the pdf and the cdf of the *first* inid OS of KW distribution when n = 3 from eq (1) and (2) in (3) and (5) using Permanent and Mathematica Program as

$$f_{1:3}(x) = a \sum_{i=1}^{3} b_i \; x^{a-1} \left[1 - x^{a}\right]^{\sum_{i=1}^{3} b_i},$$

$$x > 0,\ a > 0,\ b > 0 \quad (7)$$

$$F_{1:3}(x) = 1 - \left[1 - x^{a}\right]^{\sum_{i=1}^{3} b_i},$$

$$x > 0,\ a > 0,\ b_i > 0 \quad (8)$$

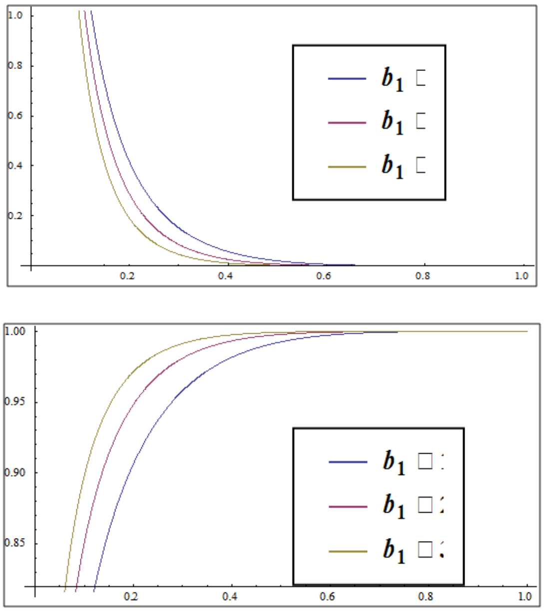

Fig. 1. Graphs of pdf and cdf of inid OS.from KW distribution for selected values of $b_2 = 1$, $b_3 = 2$, $b_1 = 1, 2, 3, 4, 5$, $n = 5, a = 0.5$

V. Moments of OS From inid Kumaraswamy rvs

Now, we consider the case when the variables $X_1, X_2, \ldots, X_n$ be inid rvs having cdfs:

$$F_i(x) = 1 - \left[1 - x^{a}\right]^{b_i}, a > 0, b_i > 0, \quad (9)$$

$$0 \le x \le 1,\ i = 1, 2, \ldots, n$$

We will use theorem of (Barakat and Abdelkader, 2003) to deduce the moments of OS from inid rvs arising from our distribution .

Theorem 1 : Let $X_1, X_2, \ldots, X_n$ be independent nonidentically distributed rvs. The k^{th} moment of all order statistics, $\mu_{r:n}^{(k)}$, for $1 \le r \le n$ and k = 1, 2,… is given by:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j(k) \quad (10)$$

Where:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < i_j \le n} \cdots \sum k \int_0^{\infty} x^{k-1} \prod_{t=1}^{j} G_{i_t}(x)\, dx \quad (11)$$

$$, j = 1, 2, \ldots, n$$

$G_{it}(x) = 1 - F_{it}(x)$ with $(i_1, i_2, \ldots, i_n)$ is a permutation of $(1, 2, \ldots, n)$ for which $i_1 \le i_2 <, \ldots, < i_n$.

The following theorem gives an explicit expression for $I_j(k)$ when $X_1, X_2, \ldots, X_n$ are OS from inid Kumaraswamy rvs.

Theorem 2: For any real numbers a, b > 0, $1 \le r \le n$ and k = 1, 2, …

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} \cdots \sum \frac{k}{a}\, \beta\left(\sum_{t=1}^{j} b_{i_t} + 1, \frac{k}{a}\right) \quad (12)$$

Proof: Applying theorem 1 and using equation (9), we get:

$$I_j(k) = k \sum_{1 \le i_1 < i_2 < i_j \le n} \cdots \sum \int_0^1 x^{k-1} \left[1 - x^{a}\right]^{\sum_{t=1}^{j} b_{i_t}} dx$$

Substituting $y = x^{a}$ the a above equation reduces to:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} \cdots \sum \frac{k}{a} \times \int_0^1 \left[1 - y\right]^{\sum_{t=1}^{j} b_{i_t}} y^{\frac{k}{a} - 1}\, dy$$

$$\therefore I_j(k) = \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} \cdots \sum \frac{k}{a} \times \beta\left(\sum_{t=1}^{j} b_{i_t} + 1, \frac{k}{a}\right)$$

where, β (a,b) is the beta function defined by:

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \beta(a,b)$$

Remark 1: The k^{th} moment of the last OS $X_{n:n}$ from inid of Kumaraswamy rvs can be written as:

$$\mu_{n:n}^{(k)} = \sum_{j=1}^{n}(-1)^{j-1} I_j(k) \qquad (13)$$

where, $I_j(k)$ is defined in (12).

Remark 2: The k^{th} moment of the first OS $X_{1:n}$ from inid of Kumaraswamy rvs can be written as:

$$\mu_{1:n}^{(k)} = I_n(k) \qquad (14)$$

Where:

$$I_n(k) = \frac{k}{a}\,\beta\left(\sum_{i=1}^{n} b_i + 1, \frac{k}{a}\right), \quad a>0 \qquad (15)$$

Remark 3: For the independent identically distributed (iid) case , we use theorem 2 . $I_j(k)$ is be written as:

$$I_j(k) = \frac{k}{a}\binom{n}{j}\beta\left(jb+1, \frac{k}{a}\right) \qquad (16)$$

VI. Numerical Calculations for Kumaraswamy Distribution

Example 1

Let n = 3 and a =1 , b_1 = 1 , b_2 = 2 , b_3 = (1 (0.5) 4) .. **Table 1** shows the results.

Table I: represents the first two moments and the variance of the median of inid OS when n = 3 and a =1, b_1 = 1, , b_2 = 2, b_3 = (1(0.5) 4)

b_3	1	1.5	2	2.5	3	3.5	4
E(x)	0.433333	0.3943	0.366667	0.346348	0.330952	0.318998	0.309524
$E(x^2)$	0.233333	0.195904	0.171429	0.154701	0.142857	0.134225	0.127778
V (x)	0.0455556	0.0404315	0.0369841	0.0347439	0.0333277	0.0324651	0.0319728

For example in (10) Let n = 3, a =1 and $b_1=1, b_2=2, b_3=3$. the k^{th} moment of the median $X_{2:3}$ is given by:

$$\mu_{2:3}^{(k)} = \sum_{j=2}^{3}(-1)^{j-2}\binom{j-1}{1} I_j(k) = I_2(k) - 2\,I_3(k) \qquad (17)$$

From (12):

$$\begin{aligned} I_2(k) &= k\sum_{1\le i_1<i_2\le 3}\cdots\sum \beta\left(\sum_{t=1}^{2} b_{i_t}+1,k\right) \\ &= k[\beta(b_1+b_2+1,k)+\beta(b_1+b_3+1,k) \\ &\quad +\beta(b_2+b_3+1,k)] \\ &= k[\beta(4,k)+\beta(5,k)+\beta(6,k)] \end{aligned} \qquad (18)$$

$$\begin{aligned} I_3(k) &= k\sum_{1\le i_1<i_2<i_3\le 3}\cdots\sum \beta\left(\sum_{t=1}^{3} b_{i_1}+1,k\right) \\ &= k\,\beta(b_1+b_2+b_3+1,k) \\ &= k[\beta(7,k) \end{aligned} \qquad (19)$$

Substituting (18) and (19) in (17), $\mu_{2:3}^{(k)}$ is then can be obtained.

Table I introduces the first two moments and the variance of the median from sample size n=3 arising from ***Kumaraswamy distribution***. These computations are done when (a=1, $b_1=1, b_2=2, b_3=(1(0.5)4)$.

Example 2

Setting n = 2, a = 2, and b_1 =1 (1) 5,b_2 = 1 (1) 5, in Theorems 1 and 2, we get the following tables.

For example, when k = 1, a = 2, b_1 = 2, b_2 = 3

$$\mu_{2:2} = I_1(1) - I_2(1) \qquad (20)$$

From (12):

Table II: The mean of the last inid OS of sample size n = 2 arising from ***Kumaraswamy distribution***

b_1 \ b_2	1	2	3	4	5
1	0.8	0.742857	0.71746	0.703608	0.695083
2	0.742857	0.660317	0.621068	0.59869	0.584482
3	0.71746	0.621068	0.573293	0.545233	0.527013
4	0.703608	0.59869	0.545233	0.51316	0.491984
5	0.695083	0.584482	0.527013	0.491984	0.468557

$$I_1(1)=\frac{1}{2}\sum_{i=1}^{2}\beta\left(b_i+1,\frac{1}{2}\right)$$
$$=\frac{1}{2}\left[\beta\left(b_1+1,\frac{1}{2}\right)+\frac{1}{2}\beta\left(b_2+1,\frac{1}{2}\right)\right] \quad (21)$$
$$=\frac{1}{2}\left[\beta\left(3,\frac{1}{2}\right)+\beta\left(4,\frac{1}{2}\right)\right]$$

$$I_2(1)=\frac{1}{2}\beta\left(\sum_{t=1}^{2}b_{i_1}+1,\frac{1}{2}\right)$$
$$=\frac{1}{2}\beta\left(b_1+b_2+1,\frac{1}{2}\right) \quad (22)$$
$$=\frac{1}{2}\beta\left(6,\frac{1}{2}\right)$$

Substituting (21) and (22) in (20), $\mu_{2:2}^{(k)}$ is then can be obtained.

Table III: The second moment of the last inid OS of sample size n = 2 arising from ***Kumaraswamy distribution***

b_1 \ b_2	1	2	3	4	5
1	0.666667	0.583333	0.55	0.533333	0.52381
2	0.583333	0.466667	0.416667	0.390476	0.375
3	0.55	0.416667	0.357143	0.325	0.305556
4	0.533333	0.390476	0.325	0.288889	0.266667
5	0.52381	0.375	0.305556	0.266667	0.242424

Table IV: The variance of the last inid OS of sample size n = 2 arising from ***Kumaraswamy distribution***

b_1 \ b_2	1	2	3	4	5
1	0.0266667	0.0314966	0.0352507	0.0382698	0.0406696
2	0.0314966	0.0306475	0.0309414	0.0320462	0.0333806
3	0.0352507	0.0309414	0.0284776	0.0277215	0.027813
4	0.0382698	0.0320462	0.0277215	0.0255557	0.024618
5	0.0406696	0.0333806	0.027813	0.024618	0.022879

Tables II, III and IV represents the mean, second moment and the variance of the last OS of sample size n = 2 arising from ***Kumaraswamy distribution***. These computations are done when (n=2, a=1, and b_1=1 (1) 5,b_2 = 1(1)5).

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