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Realization Results on Doubly Geodetic Number

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***Abstract* – For a connected graph G, a set S of vertices of G is called a doubly geodetic set of G if each vertex in $V - S$ lies on at least two distinct geodesics of vertices in S. The doubly geodetic number $\ddot{d}g(G)$ of G is the minimum cardinality of a doubly geodetic set. Any doubly geodetic set of cardinality $\ddot{d}g(G)$ is called $\ddot{d}g$-set of G. In this paper, we solve realization problems concerning geodetic number, doubly geodetic number, order, radius and diameter of a graph.**

***Keywords* – Doubly Geodetic Set, Doubly Geodetic Number, Geodetic Set, Geodetic Number.**

I. INTRODUCTION

Geodetic number of a graph is one of the widely studied graph theoretic parameters concerning geodesic convexity in graphs. Harary et al introduced this graph theoretical parameter in [2] and further studied it in [3-5, 7]. The edge geodetic number of graph was introduced in [1] and was further studied by several others. The geodetic concepts have many interesting applications in convexity theory, location theory, communication network design and designing the route for a shuttle. It is notable that the edge geodetic set has more practical applications than the geodetic sets. In particular, they are more advantageous in the case of routing and regulating the goods vehicles to transport the goods to important places. The geodetic concepts are also applied to other areas like neural network, distributed computing, facility location, data mining, telephone switching centers, image and video editing. Many variants of geodetic number have been defined and extensively studied in literature.

By a graph $G = (V, E)$, we mean a finite undirected connected simple graph with vertex set $V = V(G)$ and the edge set $E = E(G)$. The order and size of G are $|V|$ and $|E|$, respectively. For vertices x and y in G, the distance $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ geodesic. A vertex z is said to lie on an $x - y$ geodesic P if z is a vertex of P including the vertices x and y. The eccentricity $e(x)$ of a vertex x is defined by $e(x) = max\{d(x, y) : y \in V\}$. The minimum and the maximum eccentricity among vertices of G is its radius r and diameter d, respectively. The geodetic closure of the set $S \subset V(G)$ is $S^c = \{v \in V: (\exists x, y \in S)\ v \text{ is in some } x - y \text{ geodesic}\}$. The geodetic number of a graph G is defined as $g(G) = min\{|S| : S \subset V \text{ and } S^c = V\}$.

An equivalent definition for geodetic number of a graph G is given in [4] as follows, Let $I(x, y)$ be the set (interval) of all vertices lying on some $x - y$ geodesic of G, and for a nonempty subset S of $V(G)$, $I(S) = \cup_{x,y \in S} I(x, y)$. The set S of vertices of G is called a geodetic set in G if $I(S) = V$, and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number $g(G)$. In [7], it is shown that the problem of determining the geodetic number of a graph is an NP-hard problem. For graph theoretic notation and terminology, we follow [2, 6].

II. DOUBLY GEODETIC NUMBER OF A GRAPH

In [8], a new variant of geodectic number, namely doubly geodetic number of a graph was introduced and studied. In this section we recall the definition and certain results of doubly geodetic set.

Let G be graph with at least two vertices. A vertex x is said to be geodominated by the pair $\{u, v\}$ if x lies on some $u - v$ geodesic. The geodetic interval $I[u, v]$ consists of u, v together with all vertices geodominated by the pair $\{u, v\}$. Let $|I[u, v]|$ denote the number of vertices in $I[u, v]$. If $|I[u, v]| = d(u, v) + 1$, then there exist a unique $u - v$ geodesic. If $|I[u, v]| \geq d(u, v) + 1$, then there exists more than one geodesics between u and v. Let $g_p(u, v)$ be a $u - v$ geodesic and $I[g_p(u, v)]$ consists of u, v together with all vertices in $g_p(u, v)$. Its clear that $|I[g_p(u, v)]| = d(u, v) + 1$. We say that two geodesics $g_p(a, b)$ and $g_q(u, v)$ are distinct if $I[g_p(a, b)] \neq I[g_q(u, v)]$.

A set S of vertices of G is called a doubly geodetic set of G if each vertex in $V - S$ lies on at least two distinct geodesics of vertices in S. The doubly geodetic number of $\ddot{d}g(G)$ is minimum cardinality of a doubly geodetic set. Any doubly geodetic set of cardinality $\ddot{d}g(G)$ is called $\ddot{d}g$-set of G. In other words, in the $I[S]$ of the doubly geodetic set S of G the vertices of $V - S$ should occur at least twice.[8]

The following are few interesting results of doubly geodetic number of a graph.

Theorem 2.1.[8] For any graph G, $2 \leq g(G) \leq \ddot{d}g(G) \leq n$.

Theorem 2.2.[8] Every $\ddot{d}g$-set of a graph contains its extreme vertices.

Theorem 2.3.[8] Let G be a connected graph with a cut vertex v. Then each doubly geodetic set contains at least one vertex from each component of $G - v$.

Theorem 2.4.[8] Let G be a graph satisfying the following two conditions: (i) $g(G) = 2$. (ii) Let $\{u, v\}$ be the geodetic set of G. For every $u - v$ geodesic say $(u, x_1, x_2, \dots, x_{D-1}, v)$ there exist another distinct $u - v$

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geodesic say $(u, y_1, y_2, \ldots, y_{D-1}, v)$ such that $x_i \neq y_i, x_j = y_j$ and $x_k \neq y_k$ for $1 \leq i < j < k \leq D-1$. Then, $\ddot{d}g(G) = 2$.

Theorem 2.5.[8] Let G be a graph satisfying the following two conditions: (i) $g(G) = 2$. (ii) Let $\{u, v\}$ be the geodetic set of G. For every $u - v$ geodesic say $(u, x_1, x_2, \ldots, x_{D-1}, v)$ there exist another distinct $u - v$ geodesic say $(u, y_1, y_2, \ldots, y_{D-1}, v)$ such that (x_i, y_{i+1}) and $(x_{i+1}, y_i) \in E(G)$ for some i. Then, $\ddot{d}g(G) = 2$.

III. Relalization Results

Theorem 3.1. For positive integers n and k such that $3 \leq k < n$. There exist a graph G of order n and $g(G) = \ddot{d}g(G) = k$.

Proof: Let $P_{n-r+1} : u_0, u_1, u_2, \ldots, u_{n-k+1}$ be a path of order $n - k + 2$. Add $k - 2$ new vertices $v_1, v_2, v_3, \ldots, v_{k-2}$ to P_{n-r+1} by joining each vertices $v_i, 1 \leq i \leq k-2$ to the vertex u_1 and obtain the graph G of Fig. 1. Then G has order n. Moreover, the set $S = \{u_0, u_{n-k+1}, v_1, v_2, v_3, \ldots, v_{k-2}\}$ is a geodetic set and $\ddot{d}g$-set of G and so G has the desired properties.

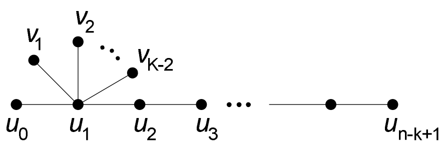

Fig. 1

Theorem 3.2. For positive integers r, d and $k \geq 3$ with $r < d \leq 2r$, there exists a graph G with $radG = r$, $diam\ G = d$, $g(G) = \ddot{d}g(G) = k$.

Proof: When $r = 1$ and $d = 1$, we have $G = K_k$ with $g(G) = \ddot{d}g(G) = k$. When $r = 1$ and $d = 2$, we have $G = K_{1,k}$ with $g(G) = \ddot{d}g(G) = k$. For $r \geq 2$ we construct a graph G with desired properties as follows. Let $C_{2r} : v_1, v_2, \ldots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, \ldots, u_{d-r}$ be a path of order $d - r + 1$. Let H be a graph obtained form C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Now, add $k - 2$ new vertices $w_1, w_2, w_3, \ldots, w_{k-2}$ to H by joining each vertices $w_i, 1 \leq i \leq k-2$ to the vertex u_{d-r-1} and obtain the graph G of Fig. 2. Then, $rad(G) = r$ and $diam(G) = d$. Moreover, it is evident that the set $S = \{v_{r+1}, u_{d-r}, w_1, w_2, w_3, \ldots, w_{k-2}\}$ is a geodetic set and $\ddot{d}g$-set of G and so G has the desired properties.

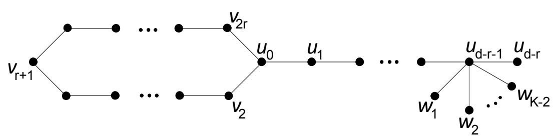

Fig. 2

Theorem 3.3. If n, d and k are integers such that $3 \leq d < n, 3 \leq k < n$ and $n - d - k + 1 \geq 0$, then there exists a graph G of order n, $diam\ G = d$ and $g(G) = \ddot{d}g(G) = k$.

Proof: Let $P_d : u_0, u_1, u_2, \ldots, u_d$ be a path of order $d + 1$. Add $k - 3$ new vertices $v_1, v_2, v_3, \ldots, v_{k-3}$ to P_d by joining each vertices $v_i, 1 \leq i \leq k-3$ to the vertex u_1, and then add a vertex x to u_{d-1} producing a tree T. Now, add $n - d - k + 1$ new vertices $w_1, w_2, w_3, \ldots, w_{n-d-k+1}$ to T by joining each vertices $w_i, 1 \leq i \leq n - d - k + 1$ to both u_0 and u_2, obtaining the graph G of Fig. 3. Then, G has order n and diameter d. Moreover, it is evident that the set $= \{u_0, u_d, x, v_1, v_2, v_3, \ldots, v_{k-3}\}$ is a geodetic set and $\ddot{d}g$-set of G and so G has the desired properties.

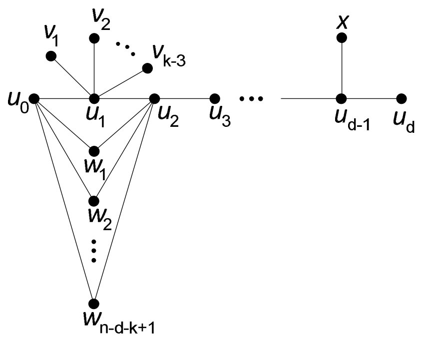

Fig. 3

Theorem 3.4. For positive integers r, d, p and k such that with $r < d \leq 2r$, $3 < p \leq k$ and $r > 1$ there exists a graph G with $radG = r$, $diam\ G = d$, $g(G) = p$ and $\ddot{d}g(G) = k$.

Proof: Case i: When $r = 2$ and $d = 3$, we construct a graph G with desired properties as follows. Let $P_2 : u_0, u_1, u_2$ be a path of order 3. Add $p - 2$ new vertices $v_1, v_2, v_3, \ldots, v_{p-2}$ to P_2 by joining each vertices $v_i, 1 \leq i \leq p-2$ to the vertex u_1, producing a tree T. Now, add $k - p + 2$ new vertices $w_1, w_2, w_3, \ldots, w_{k-p+2}$ to T by joining each vertices $w_i, 1 \leq i \leq k - p + 2$ to both $u_0$and u_2, obtaining the graph G of Fig. 4. Then, G has radius 2 and diameter 3. Clearly, the geodetic set and doubly geodetic set of G are $S = \{u_0, u_2, v_1, v_2, v_3, \ldots, v_{p-2}\}$ and $S' = \{v_1, v_2, v_3, \ldots, v_{p-2}, w_1, w_2, w_3, \ldots, w_{k-p+2}\}$ respectively. Thus $g(G) = p$ and $\ddot{d}g(G) = k$.

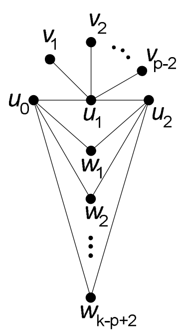

Fig. 4

Case ii: When $r = 2$ and $d = 4$, we construct a graph G with desired properties as follows. Let $\overline{K_2} : x, y$ be the complement of K_2. Add $p - 2$ new vertices $u_0, u_1, u_2, \ldots, u_{p-2}$ to $\overline{K_2}$ by joining each vertices $u_i, 1 \leq i \leq p-2$ to both x and y and obtain the graph H. Then, add $p-2$ new vertices $v_1, v_2, v_3, \dots, v_{p-2}$ to H by joining each vertices v_i to each vertex u_i, $1 \le i \le p-2$ and obtain the graph H'. Now, add $k-p+2$ new vertices $w_1, w_2, w_3, \dots, w_{k-p+2}$ to H' by joining each vertices $w_i, 1 \le i \le k-p+2$ to both x and y, obtaining the graph G of Fig. 5. Then, G has radius 2 and diameter 4. Clearly, the sets $S = \{x, y, v_1, v_2, v_3, \dots, v_{p-2}\}$ and $S' = \{v_1, v_2, v_3, \dots, v_{p-2}, w_1, w_2, w_3, \dots, w_{k-p+2}\}$ are the geodetic set and doubly geodetic set of G with $g(G) = p$ and $\ddot{d}g(G) = k$, respectively.

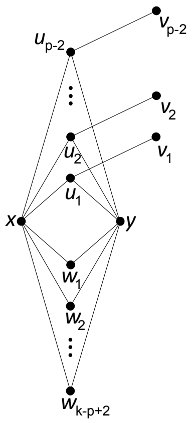

Fig. 5

Case iii: When $r \ge 3$, we construct graph G with desired properties as follows. Let C_{2r-2}: $x_1, x_2, x_3, \dots, x_{r-1}, x_r, x_{r+1}, \dots, x_{2r-2}, x_1$ be a cycle of order $2r-2$ and let $P_{d-r}: u_0, u_1, u_2, \dots, u_{d-r}$ be a path of order $d-r+1$. Let H be a graph obtained form C_{2r} and P_{d-r} by identifying x_r in C_{2r} and u_0 in P_{d-r}. Add $p-3$ new vertices $v_1, v_2, v_3, \dots, v_{p-3}$ to H by joining each vertices $v_i, 1 \le i \le p-3$ to the vertex x_1. Now, add $k-p+2$ new vertices $w_1, w_2, w_3, \dots, w_{k-p+2}$ and join with both x_2 and x_{2r-2} and obtain the graph G of Fig. 6. Then, $rad(G) = r$ and $diam(G) = d$. It is evident that the sets $S = \{x_2, x_{2r-2}, u_{d-r}, v_1, v_2, v_3, \dots, v_{p-3}\}$ and $S' = \{u_{d-r}, v_1, v_2, v_3, \dots, v_{p-3}, w_1, w_2, w_3, \dots, w_{k-p+2}\}$ are the geodetic set and the doubly geodetic set of G, with $g(G) = p$ and $\ddot{d}g(G) = k$, respectively.

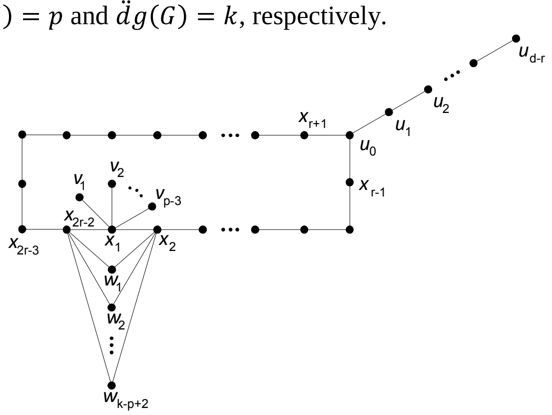

Fig. 6

Theorem 3.5. For positive integers n, d, p and k such that $d \ge 6, 3 \le p \le k \le n$, $k-p+2 \ge 0$ and $n-d-k \ge 0$. There exist a graph G of order n, $\text{diam}(G) = \text{d}$, $g(G) = p$ and $\ddot{d}g(G) = k$.

Proof: Let $P_d: u_0, u_1, u_2, \dots, u_d$ be a path of order $d+1$. Add $p-3$ new vertices $v_1, v_2, v_3, \dots, v_{p-3}$ to P_d by joining each vertices $v_i, 1 \le i \le p-3$ to the vertex u_1, producing a tree T. Then, add $k-p+2$ new vertices $w_1, w_2, w_3, \dots, w_{k-p+2}$ to T by joining each vertices $w_i, 1 \le i \le k-p+1$ to both u_0 and u_2 obtaining the graph H. Add $n-d-k$ new vertices $x_1, x_2, \dots, x_{n-d-k}$ to H by joining each vertices $x_i, 1 \le i \le n-d-k$ to both u_4 and u_6 in H forming the graph H'. The graph G is obtainied by adding an edge $u_0 - u_3$ to H'. See Fig. 7.

Then, G has order n and diameter d. Moreover, it is evident that the set $S = \{u_0, u_2, u_d, v_1, v_2, v_3, \dots, v_{p-3}\}$ is a geodesic set with $g(G) = p$ and the set $S' = \{u_d, v_1, v_2, \dots, v_{p-3}, w_1, w_2, w_3, \dots, w_{k-p+2}\}$ is a $\ddot{d}g$-set of G with $\ddot{d}g(G) = k$.

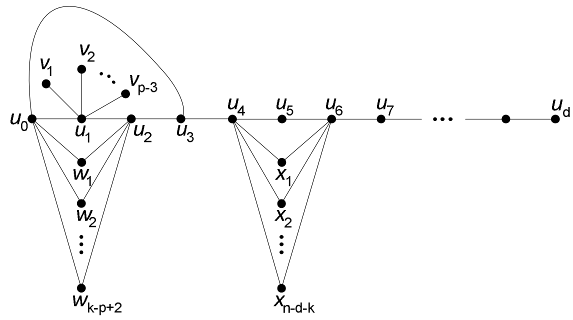

Fig. 7

IV. Conclusion

In this paper, we have studied certain realization problems involving geodetic number, doubly geodetic number, order and diameter of a graph. It is also shown that for every pair k, n of integers with $3 \le k < n$ is realizable as the doubly geodetic number and order of some connected graph. Our further research is to investigate the doubly geodetic problem for graph products and various graph families

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