International Journal of Innovation in Science and Mathematics
Volume 2, Issue 1, ISSN (Online): 2347–9051

On the Comparative Study of Some Numerical Methods for the Solution of Initial Value Problems in Ordinary Differential Equations

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***Abstract* – This paper presents some numerical methods for the solution of initial value problems in ordinary differential equations namely; Runge Kutta method, Euler's method and an implicit linear multistep method of order six. Runge Kutta method attempts to obtain greater accuracy and at the same time avoids the need for higher derivatives by evaluating the given function at selected points on each subinterval. Euler's method is presented from the point of view of Taylor's algorithm which considerably simplifies the rigorous analysis. The derivation of a six step method is based on the interpolation and collocation methods. Numerical experiments were taken into consideration to determine the accuracy of the methods. An implicit linear multistep method of order six is more accurate and converges faster than Runge Kutta method and Euler's method.**

***Keywords* – Convergence, Differential Equation, Error, Euler's Method, Implicit Linear Multistep Method, Runge Kutta Method**

***2010 Mathematics Subject Classification:* 34A12, 34A26, 33F10**

I. Introduction

Historically, it has been discovered that mathematical models resulting into single or system of first order ordinary differential equations are largely applied in nearly all disciplines most especially in Sciences, Engineering and Economics. Differential equations are one of the most important mathematical tools used in modeling problems in physical sciences. However, many systems involving differential equations are so complex. It is in these complex systems where computer simulations and numerical approximations are useful. The techniques for solving differential equations based on numerical approximations were developed before computer programming existed. The problem of the solution of ordinary differential equations is classified into initial value problem and boundary value problem, depending on the conditions given at the end points of the domain. There are numerous methods that produce numerical approximations to solution of initial value problems in ordinary differential equations such as Euler's method which was the oldest and simplest method originated by Leonhard Euler in 1768, Improved Euler's method, Adam Bashforth, Adam Moulton and Runge Kutta methods described by Carl Runge and Martin Kutta in 1895 and 1905 respectively. All these methods discretize the differential system to produce a difference equation. With the advent of computers, numerical methods are now an increasingly attractive and efficient way to obtain approximate solutions to differential equations that cannot be solved analytically.

There are many excellent and exhaustive books on this subject that may be consulted, such as [3], [8], and [9] just to mention few. In this paper we present the comparative results analysis, practical use and the accuracy of Runge Kutta method, Euler's method and an implicit linear multistep method for the solution of initial value problems in ordinary differential equations.

II. Materials and Methods

This section presents Runge Kutta method, Euler's method and an implicit linear multistep method for the solution of initial value problems in ordinary differential equations as follows:

A. Runge Kutta Method

Runge Kutta method is a technique for approximating the solution of ordinary differential equation. This technique was developed around 1900 by the mathematicians Carl Runge and Wilhelm Kutta. Runge Kutta method is popular because it is efficient and used in most computer programs for differential equation.

The initial value problem given in (1) below shall be considered:

$$y' = f(x, y),\ y(x_0) = y_0 \qquad (1)$$

The Runge Kutta methods for the solution of (1) are one step methods designed to approximate Taylor series methods, but have the advantage of not requiring explicit evaluation of the derivatives of $f(x, y)$, where x represents time. The basic idea is to use a linear combination of values of $f(x, y)$ to approximate $y(x)$.

This linear combination is matched up as closely as possible with a Taylor series for $y(x)$ to obtain methods of the highest possible order p.

The Runge Kutta Method has the following orders as listed below [5]:

- First order Runge Kutta method is called Euler's method.
- Second order Runge Kutta method is called modified Euler's or Heun's Method.
- Fourth order Runge Kutta method is the same as classical Runge Kutta method.

1) Derivation of Runge Kutta Method [4, 7]

In this paper, we shall only consider the fourth order Runge Kutta method. We shall derive here the simplest of the Runge method using a formula of the form:

$$y_{n+1} = y_n + ak_1 + bk_2 \qquad (2)$$

Where $k_1 = hf(x_n, y_n)$, $k_2 = hf(x_n + \alpha h, y_n + \beta k_1)$ and a, b, α, β are constants to be determined so that (2) will agree with the Taylor algorithm. Expanding $y(x_{n+1})$ in a Taylor series of order h^3, we obtain

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2 y''(x_n)}{2} + \frac{h^3 y'''(x_n)}{6} + \ldots$$

$$= y(x_n) + hf(x_n, y_n) + \frac{h^2 (f_x + ff_y)_n}{2}$$

$$+ \frac{h^3 (f_{xx} + 2ff_{xy} + f_{yy} f^2 + f_x f_y + f_y{}^2 f)_n}{6} + 0(h^4)$$

It should be noted that the expansions $y' = f(x, y)$, $y'' = f_x + f_y f$ and $y''' = f_{xx} + 2f_{xy} f + f_{yy} f^2 + f_x f_y + f_y{}^2 f$.

The subscript n means that all functions involved are to be evaluated at (x_n, y_n).

On the other hand, using Taylor's expansion for functions of two independent variables, we have that

$$k_2 = f(x_n + \alpha h, y_n + \beta k_1 = f(x_n, y_n) + \alpha h f_x$$

$$+ \beta k_1 f_y + \frac{\alpha^2 h^2 f_{xx}}{2} + \alpha h \beta k_1 f_{xy} + \frac{\beta^2 k_1^2 f_{yy}}{2} + 0(h^3)$$

All the derivatives above are evaluated at (x_n, y_n). If we now substitute this expression for k_2 into (2) and note that $k_1 = hf(x_n, y_n)$, we find upon rearrangement in powers of h and by setting $a = b = \frac{1}{2}, \alpha = \beta = 1$ we have that,

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (3)$$

Where,

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1)$$

$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_2)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

This method in (3) is undoubtedly the most popular of all Runge Kutta methods and it is referred to as the "fourth order Runge Kutta method". Many numerical analysts use (3) for solving initial value problems in ordinary differential equations, because it is quite stable, accurate, consistent and easy to program.

2) *Error Estimate for Runge Kutta Method*

One of the serious draw backs of Runge Kutta method is error estimation. For all one step methods like Runge Kutta Method, we define a local truncation error as the terms we discard when generating a numerical scheme from Taylor expansion. We denote the solution to the initial value problem in (1) by $x, x(0), y(0)$. We have noted that the truncation error in p^{th} order Runge Kutta method is kp^{p+1}, where k is some constant. Bounds on k for $p = 2,3,4$ also exist. The derivation of these bounds is not a simple matter because they need some quantities in evaluating them [7].

Under the localizing assumption that no previous errors have been made, we may write [1]:

$$T_{n+1} = y(x_{n+1}) - y_{n+1}$$
$$= \lambda(x_n, y(x_n))h^{p+1} + o(h^{p+2})$$

where p is the order of the Runge Kutta method, $\lambda(x_n, y(x_n))h^{p+1}$ is the principal local truncation error. Next we shall compute y^*_{n+1}, a second approximation to $y(x_{n+1})$, obtained by applying the same method at x_{n-1} with step length $2h$. Then we have that

$$y(x_{n+1}) - y^*_{n+1} =$$
$$\lambda(x_{n-1}, y(x_{n-1}))(2h)^{p+1} + o(h^{p+2})$$

Therefore, the principal local truncation error that is taken as an estimate for the local truncation error may be written as:

$$\lambda(x_n, y(x_n))h^{p+1} = T_{n+1} = \frac{(y(x_{n+1}) - y^*_{n+1})}{(2^{p+1} - 1)}$$

This implies that:

$$T_{n+1} = \frac{(y(x_{n+1}) - y^*_{n+1})}{(2^{p+1} - 1)} \qquad (4)$$

(4) is a means of obtaining quick estimates of the local truncation errors in computations using any S-stage Runge Kutta Method, without h obtaining the exact solution first.

3) *Stability of Runge Kutta Method*

The *A*-stability concept for the solution of differential equations is related to the linear autonomous equation $y' = \lambda y$. Dahlquist proposed the investigation of stability of numerical schemes when they applied to a nonlinear system which satisfies a monotonicity condition. The corresponding concepts were defined as *G*-stability for multistep methods and *B*-stability for Runge Kutta methods. A Runge Kutta method applied to the nonlinear system $y' = f(y)$, which verifies $\langle f(y) - f(z), y - z \rangle < 0$, is called *B*-stable, if this condition implies $\|y_{n+1} - z_{n+1}\| \le \|y_n - z_n\|$ for two numerical solutions. Let B, N and R be three $s \times s$ matrices defined by $B = diag(b_1, b_2, ..., b_s)$, $N = BA + A^T B - bb^T$ and

$$R = BA^{-1} + A^{-T}B - A^{-T}bb^T A^{-1}$$

A Runge Kutta method is said to be algebraically stable if the matrices B and N are both non-negative definite. A sufficient for *B*-stability is B and R are non-negative definite.

B. Euler's Method

Euler's method is also called tangent line method and is the simplest numerical method for solving initial value problems in ordinary differential equations was originated by Leonhard Euler in 1768. Euler's method consists of three forms namely,

- Forward Euler's method.
- Improved Euler's method.
- Backward Euler's method.

In this paper, forward Euler's method shall be considered.

1) Derivation of Euler's method [4]

We present below the derivation of Euler's method for the solution of the initial value problem in (1), where x_0 and y_0 are initial values for x and y respectively. Our purpose is to determine (approximately) the unknown function $y(x)$ for $x \geq x_0$. We are told explicitly the value of $y(x_0)$, namely y_0, using the given differential equation in (1), we can determine the rate of change of y at point x_0

$$y'(x_0) = f(x_0, y(x_0)) = f(x_0, y_0)$$

If the rate of change of $y(x)$ were to remain $f(x_0, y_0)$ for all point x, then $y(x)$ would exactly $y_0 + f(x_0, y_0)(x - x_0)$.The rate of change of $y(x)$ does not remain close to $f(x_0, y_0)$ for all x, but it is reasonable to expect that it remains close to $f(x_0, y_0)$ for $x \approx x_0$. Then If this is the case, the value of $y(x)$ will remain close to $y_0 + f(x_0, y_0)(x - x_0)$ for $x \approx x_0$, for small number h, we have

$$x_1 = x_0 + h \quad (5)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$
$$= y_0 + hf(x_0, y_0) \quad (6)$$

Where $h = x_1 - x_0$ and is called the step size.

By the above argument,

$$y(x_1) \approx y_1 \quad (7)$$

Repeating the above process, we have at point x_1 as follows

$$x_2 = x_1 + h \quad (8)$$

$$y_2 = y_1 + f(x_1, y_1)(x_2 - x_1)$$
$$= y_1 + hf(x_1, y_1) \quad (9)$$

We have

$$y(x_2) \approx y_2 \quad (10)$$

Then define for $n = 0,1,2,3,4,5,\ldots$, we have

$$x_n = x_0 + nh \quad (11)$$

Suppose that, for some value of n, we are already computed an approximate value y_n for $y(x_n)$. Then the rate of change of $y(x)$ for x close to x_n is $f(x, y(x)) \approx f(x_n, y(x_n)) \approx f(x_n, y_n)$ where $y(x_n) = y_n + f(x_n, y_n)(x - x_n)$.

Thus,

$$y(x_{n+1}) \approx y_{n+1} = y_n + hf(x_n, y_n) \quad (12)$$

Hence,

$$y_{n+1} = y_n + hf(x_n, y_n) \quad (13)$$

(13) is called Euler's method.

From (13), we have

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n), n = 0,1,2,3,\ldots \quad (14)$$

2) Truncation Errors for Euler's Method [6]

Numerical stability and errors are discussed in depth in [3] and [8]. There are two types of errors arise in numerical methods namely truncation error which arises primarily from a discretization process and round off error which arises from the finiteness of number representations in the computer. In order to estimate the truncation error for Euler's method, we first recall Taylor's theorem with remainder, which states that a function $f(x)$ can be expanded in a series about the point $x = a$

$$f(x) = f(a) + f'(a)(x-a)\frac{f''(a)(x-a)^2}{2!} + \ldots + \frac{f^m(a)(x-a)^m}{m!} + \frac{f^{m+1}(\beta)(x-a)^{m+1}}{(m+1)!} \quad (15)$$

The last term of (15) is referred to as the remainder term. Where $x \leq \beta \leq a$.

In (15), let $x = x_{n+1}$ and $x = a$, in which

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{1}{2}h^2 y''(\beta_n) \quad (16)$$

Since y satisfies the ordinary differential equation in (1), which can be written as

$$y'(x_n) = f(x_n, y(x_n)) \quad (17)$$

Hence,

$$\left.\begin{aligned} y(x_{n+1}) &= y(x_n) + hf(x_n, y(x_n)) \\ &+ \frac{1}{2}h^2 y''(\beta_n) \end{aligned}\right\} \quad (18)$$

By comparing (18) with Euler's approximation in (13), it is clear that Euler's method is obtained by omitting the remainder term $\frac{1}{2}h^2 y(\beta_n)$ in the Taylor expansion of $y(x_{n+1})$ at the point x_n. The omitted term accounts for the truncation error in Euler's method at each step.

3) Convergence of Euler's Method [4, 6]

The necessary and sufficient conditions for a numerical method to be convergent are stability and consistency. Stability deals with growth or decay of error as numerical computation progresses. Now we state the following theorem to discuss the convergence of Euler's method.

Theorem: If $f(x,y)$ satisfies a Lipschitz condition in y and is continuous in x for $0 \le x \le a$ and defined a sequence y_n, where $n = 1,2,\ldots,k$ and if $y_0 \to y(0)$, then $y_n \to y(x)$ as $n \to \infty$ uniformly in x where $y(x)$ is the solution of (1).

Proof: we shall begin the proof of the above theorem by deriving a bound for the error

$$e_n = y_n - y(x_n) \tag{19}$$

Where y_n and $y(x_n)$ are called numerical and exact values respectively. We shall also show that this bound can be made arbitrarily small. If a bound for the (19) depends only on the knowledge of the problem but not on its solution $y(x)$, it is called an a priori bound. Otherwise, it is called an a posteriori bound.

To get an a priori bound, let us write

$$y(x_{n+1}) = y(x_n) + hf(x_n, y_n) - t_n \tag{20}$$

Where x_n is called the local truncation error which is defined as the quantity by which the solution fails to satisfy the difference method. Subtracting (20) from (13), we get

$$e_{n+1} = e_n + h[f(x_n, y_n) - f(x_n, y(x_n))] + t_n \tag{21}$$

Let us write

$$e_n M_n = f(x_n, y_n) - f(x_n, y(x_n)) \tag{22}$$

Substituting (21) into (22), then

$$e_{n+1} = e_n(1 + hM_n) \tag{23}$$

This is the difference equation for e_n. The error e_0 is known, so it can be solved if we know M_n and t_n. We have a bound of the Lipschitz constant M for $|M_n|$.

Suppose we also have $T \ge |t_n|$. Then we have

$$|e_{n+1}| \le |e_n|(1 + hM) + T \tag{24}$$

To continue the proof, we shall state following lemma.

Lemma**:** If $|e_n|$ satisfies (24) and $0 \le nh \le a$, then

$$|e_n| = T\frac{(1+hM)^n - 1}{hM} + (1+hM)^n|e_0|$$
$$\le \frac{T}{hM}(e^{Lb} - 1) + e^{Lb}|e_0| \tag{25}$$

Lemma: The first inequality of (25) follows by induction. It is trivially true for $n = 0$. Assuming that it is true for n, we have from (24)

$$\left.\begin{aligned} |e_{n+1}| &= T\frac{(1+hM)^n - 1}{hM}(1+hM)^n \\ &+ T + (1+hM)^{n+1}|e_0| \\ &= T\frac{(1+hM)^{n+1} - (1+hM) + hM}{hM} \\ &+ (1+hM)^{n+1}|e_0| \end{aligned}\right\} \tag{26}$$

Hence (25) is true for $n+1$ and thus for all n.

The second inequality in (25) follows from the fact that $nh \le a$ and for $hM \ge 0$, $(1+hM) \le e^{Mh}$ so that $(1+hM)^n \le e^{Mnh} \le e^{Ma}$, proving the lemma.

To continue the proof of the theorem, we need to investigate T, the bound on the local truncation error.

From (20), we have

$$-t_n = y(x_{n+1}) - y(x_n) - hf(x_n, y(x_n))$$

By the Mean value theorem, we get for $0 \le \theta \le 1$,

$$\left.\begin{aligned} &\le h|f(x_n + \theta h, y(x_n)) - f(x_n, y(x_n))| + \\ &h|f(x_n + \theta h, y(x_n + \theta h)) - f(x_n + \theta h, y(x_n))| \\ &\le h|f(x_n + \theta h, y(x_n)) - f(x_n, y(x_n))| \\ &+ h|y(x_n + \theta h) - y(x_n)| \end{aligned}\right\} \tag{27}$$

Using the Mean value theorem for the treatment of the last term to get a bound $M\theta h^2|y'(g)| \le h^2 MZ$, where $Z = \max|y'(x)|$, the inequality exists because of the continuity of y and f in a closed region. The treatment of the first term in (27) depends on our hypothesis. If we are prepared to assume that $f(x,y)$ also satisfies a Lipschitz condition in x, we can bound the first term in (27) by $L\theta h^2$, where L is the Lipschitz constant for $f(x)$. Consequently, $|t_n| \le h^2(L + MZ) = T$ and so from (25), we get

$$|e_n| \le h\frac{L + MZ}{M}(e^{Ma} - 1) + e^{Ma}|e_0| \tag{28}$$

Thus the numerical solution converges as $h \to 0$, if $|e_0| \to 0$.

C. Linear Multistep Method

Linear multistep method is a numerical method whereby a numerical approximation y_{n+1} to the exact solution $y(x_{n+1})$ of the first order initial value problem of the form (1).

The general linear multistep method is of the form [9]

$$\sum_{k=0}^{j} \alpha_k y_{n+k} = h\sum_{k=0}^{j} \beta_k f_{n+k} \tag{29}$$

Where α_k and β_k are constants, h is the step size. It is assumed that the function $f(x,y)$ is Lipschitz continuous throughout the interval $a \le x \le b$. (29) includes Simpson method, Adam Bashforth and Adam Molton methods. All Adam's methods are regarded as constant coefficient method but in this paper, linear multistep method with constant coefficient of higher step number j is generated. The parameters of this method are determined by the collocation approach in which the approximate solution is determined from the condition that the equation must be stratified at certain given point. It involves the determination of an approximate solution in a suitable set of function called the basis function.

International Journal of Innovation in Science and Mathematics
Volume 2, Issue 1, ISSN (Online): 2347–9051

Now we shall derive an order six implicit linear multistep method for the solution of first order differential equation using collocation and interpolation methods.

Collocation points are used to arrange different systems. The interpolation points are used to interpolate the approximate solution with the Figure1 below. Both Collocation and interpolation are done at all even points $x = x_n, x_{n+2}$ and x_{n+4} while evaluation is done at $x = x_{n+6}$.

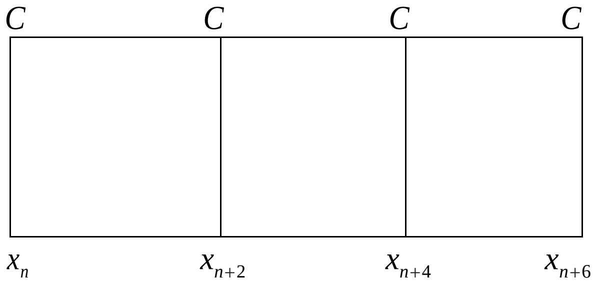

Fig.1. Evaluation Points

In the sequel, the derivation of the implicit scheme shall be presented as follows.

1) Derivation of the Implicit Scheme [2, 10]

The basis function is given by:

$$y(x) = \sum_{k=0}^{6} b_k x^k \tag{30}$$

is needed in the derivation of the scheme for solving first order differential equation.

Expanding (30) we have,

$$\left. \begin{aligned} y(x) &= b_0 + b_1 x + b_2 x^2 \\ &\quad + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \end{aligned} \right\} \tag{31}$$

Differentiating (31) with respect to x

$$\left. \begin{aligned} y'(x) &= b_1 + 2b_2 x + 3b_3 x^2 \\ &\quad + 4b_4 x^3 + 5b_5 x^4 + 6a_6 x^5 \end{aligned} \right\} \tag{32}$$

Collocating (32), we have that at $x = x_n, x_{n+2}, x_{n+4}$ and x_{n+6}

Therefore,

$$\begin{aligned} f_n &= b_1 + 2b_2 x_n + 3b_3 x_n^2 \\ &\quad + 4b_4 x_n^3 + 5b_5 x_n^4 + 6b_6 x_n^5 \end{aligned} \tag{33}$$

$$\left. \begin{aligned} f_{n+2} &= b_1 + 2b_2 x_{n+2} + 3b_3 x_{n+2}^2 \\ &\quad + 4b_4 x_{n+2}^3 + 5b_5 x_{n+2}^4 + 6b_6 x_{n+2}^5 \end{aligned} \right\} \tag{34}$$

$$\left. \begin{aligned} f_{n+4} &= b_1 + 2b_2 x_{n+4} + 3b_3 x_{n+4}^2 \\ &\quad + 4b_4 x_{n+4}^3 + 5b_5 x_{n+4}^4 + 6b_6 x_{n+4}^5 \end{aligned} \right\} \tag{35}$$

$$\left. \begin{aligned} f_{n+6} &= b_1 + 2b_2 x_{n+6} + 3b_3 x_{n+6}^2 \\ &\quad + 4b_4 x_{n+6}^3 + 5b_5 x_{n+6}^4 + 6b_6 x_{n+6}^5 \end{aligned} \right\} \tag{36}$$

Interpolating at the points $x = x_n, x_{n+2}$ and x_{n+4}, then

$$\left. \begin{aligned} y_n &= b_0 + b_1 x_n + b_2 x_n^2 \\ &\quad + b_3 x_n^3 + b_4 x_n^4 + b_5 x_n^5 + b_6 x_n^6 \end{aligned} \right\} \tag{37}$$

$$\left. \begin{aligned} y_{n+2} &= b_0 + b_1 x_{n+2} + b_2 x_{n+2}^2 \\ &\quad + b_3 x_{n+2}^3 + b_4 x_{n+2}^4 + b_5 x_{n+2}^5 + b_6 x_{n+2}^6 \end{aligned} \right\} \tag{38}$$

$$\left. \begin{aligned} y_{n+4} &= b_0 + b_1 x_{n+4} + b_2 x_{n+4}^2 \\ &\quad + b_3 x_{n+4}^3 + b_4 x_{n+4}^4 + b_5 x_{n+4}^5 + b_6 x_{n+4}^6 \end{aligned} \right\} \tag{39}$$

Using Gaussian Elimination method to determine the values of the coefficients $b_{0,} b_1, b_2, b_3, b_4, b_5$ and b_6, then we have that:

$$\left. \begin{aligned} b_0 &= y_n - b_1 x_n - b_2 x_n^2 \\ &\quad - b_3 x_n^3 - b_4 x_n^4 - b_5 x_n^5 - b_6 x_n^6 \end{aligned} \right\} \tag{40}$$

$$\left. \begin{aligned} b_1 &= f_n - 2b_2 x_n - 3b_3 x_n^2 \\ &\quad - 4b_4 x_n^3 - 5b_5 x_n^4 - 6b_6 x_n^5 \end{aligned} \right\} \tag{41}$$

$$\left. \begin{aligned} b_2 &= \frac{1}{4h^2}\left(y_{n+2} - y_n\right) - \frac{1}{2h} f_n \\ &-b_3\left(3x_n + 2h\right) - b_4\left(6x_n^2 + 8x_n h + 4h^2\right) \\ &-b_5\left(10x_n^3 + 20x_n^2 + 2o x_n h + 8h^3\right) \\ &-b_6\left(15x_n^4 + 40x_n^3 h + 60x_n^2 h + 48x_n h^3 + 16h^4\right) \end{aligned} \right\} \tag{42}$$

$$\left. \begin{aligned} b_3 &= \frac{1}{4h^2}\left(f_{n+2} + f_n\right) - \frac{1}{4h^3}\left(y_{n+2} - y_n\right) \\ &-b_4\left(4x_n + 4h\right) - b_5\left(10x_n^2 + 20x_n h + 12h^2\right) \end{aligned} \right\} \tag{43}$$

$$\left. \begin{aligned} b_4 &= \frac{1}{64h^4}\left(y_{n+4} + 4f_{n+2} - 5y_n\right) \\ &-\frac{1}{16h^3}\left(2f_{n+2} + f_n\right) - b_5\left(5x_n + 8h\right) \\ &-b_6\left(15x_n^2 + 48x_n h + 44h^2\right) \end{aligned} \right\} \tag{44}$$

$$\left. \begin{aligned} b_5 &= \frac{1}{64h^4}\left(y_{n+4} + 4f_{n+2} + f_n\right) \\ &\quad - \frac{3}{128h^5}\left(y_{n+4} - y_n\right) \\ &\quad - b_6\left(6x_n + 12h\right) \end{aligned} \right\} \tag{45}$$

$$\left. \begin{aligned} b_6 &= \frac{1}{4224h^5}\left(f_{n+6} - 24f_{n+4} - 57f_{n+2} - 10f_n\right) \\ &+ \frac{1}{2816h^5}\left(19f_{n+4} - 8f_{n+2} - 11f_n\right) \end{aligned} \right\} \tag{46}$$

By inserting the coefficient b_0 into (30), evaluating at $x = x_{n+6}$, substituting (41), (42), (43), (44), (45) and (46) for b_1, b_2, b_3, b_4, b_5 and b_6 respectively and simplify we have the scheme:

$$\left.\begin{aligned} &y_{n+6} + \frac{27}{11} y_{n+4} - \frac{27}{11} y_{n+2} - y_n \\ &= \frac{6}{11} h\left(f_{n+6} + 9 f_{n+4} + f_n \right) \end{aligned}\right\} \quad (47)$$

is called an implicit linear multistep method of order six.

III. Numerical Experiments and Results

Numerical experiments are mentioned to prove which numerical methods converge faster to analytic solutions.

In order to confirm the degree of relevancy and suitability of the Runge Kutta, Euler's and implicit linear methods for the solution of initial value problem in ordinary differential equations, it was computerized in Q BASIC programming languages and implemented on a macro-computer adopting double precision arithmetic.

The performance of the three methods under consideration was checked by comparing their accuracy and efficiency. Efficiency was determined from the number iterations counts and the number of the functions evaluations per step while the accuracy is determined by the size of the discretization errors estimated from the difference between the true solution and the numerical approximations.

We shall consider first order initial value problem given by

$$y' = -y,\ y(0) = 1,\ x \in [0,1] \quad (48)$$

The true solution of equation (24) is given by

$$y(x) = e^{-x} \quad (49)$$

The results obtained shown in Tables 1 and 2, the comparison of the three methods to the true solution and the error incurred respectively.

A. Table of Results

We present below the comparative result analysis and the error incurred from the two methods in Tables 1 and 2 respectively.

B. Discussion of Results

As we can see from the above Table 1, using a step size of 0.1, implicit linear multistep method converges faster than the other two methods. Also from Table 2, the error incurred in an implicit linear multistep method is smaller than that of its counterparts Runge Kutta method and Euler's method. Hence a six step implicit method is consistent and better in accuracy.

IV. Conclusion

In this paper, some numerical methods for the solution of initial value problems in ordinary differential equations have been developed. Each of the numerical methods has its own advantages and disadvantages of use. Runge Kutta method is good choice to get more accurate and more efficient solutions for solving the specified ordinary differential equations. The approximated solution converges faster to exact solution and the order of classical Runge Kutta method is 4 and the truncation error is $O(h^5)$. The derivation of Runge-Kutta method is obtained from Taylor series, but it is tedious to calculate higher derivative. Euler's method is the simplest of all linear multi-step method to obtain the approximated solution of the specified initial value problem, it has the one-step techniques and it can be easily programmed but in spite of the simplicity, it is restricted to use because it generate large error in each successive step during the computation which is the accumulated error and has slow rate of convergence. Implicit linear multistep method approximate numerical values of the solution by referring to more than one previous value, converges faster, it has smaller error constants and larger stability regions, it uses fewer steps. Accordingly, this method may often achieve greater accuracy than one-step methods that use the same number of function evaluations, since they utilize more information about the known portion of the solution than one-step methods do, but it is computationally expensive. From the three methods considered, an implicit linear multistep method of order six converges faster, provides the closest accurate value for the solution of any first order differential equations. Hence the implicit linear multistep method is more accurate than the other two methods namely; Runge Kutta and Euler's methods.

Appendix

Table 1: Comparative Result Analysis of Runge Kutta Method, Euler Method and an Implicit Linear Multistep Method

n	x_n	True Solution $y(x_n)$	Runge Kutta Method y_{nR}	Euler's Method y_{nE}	Implicit Linear Multistep Method y_{nI}
0	0.0	1.0000	1.0000	1.0000	1.0000
1	0.1	0.9048	0.9045	0.9097	0.9047
2	0.2	0.8187	0.8185	0.8275	0.8186
3	0.3	0.7408	0.7404	0.7526	0.7407
4	0.4	0.6703	0.6701	0.6845	0.6703
5	0.5	0.6065	0.6064	0.6226	0.6065
6	0.6	0.5488	0.5485	0.5662	0.5487
7	0.7	0.4965	0.4966	0.5068	0.4965
8	0.8	0.4493	0.4491	0.4682	0.4493

9	0.9	0.4066	0.4065	0.4257	0.4067
10	1.0	0.3678	0.3679	0.3871	0.3678

Table 2: Error incurred in Runge Kutta Method, Euler's Method and Implicit Linear Multistep Method

n	x_n	$e_{nR} = \lvert y(x_n) - y_{nR} \rvert$	$e_{nE} = \lvert y(x_n) - y_{nE} \rvert$	$e_{nI} = \lvert y(x_n) - y_{nI} \rvert$
0	0.0	0.0000	0.0000	0.0000
1	0.1	0.0002	0.0005	0.0001
2	0.2	0.0002	0.0009	0.0001
3	0.3	0.0004	0.0118	0.0001
4	0.4	0.0002	0.0142	0.0000
5	0.5	0.0001	0.0161	0.0000
6	0.6	0.0003	0.0174	0.0001
7	0.7	0.0001	0.0103	0.0000
8	0.8	0.0002	0.0189	0.0000
9	0.9	0.0001	0.0191	0.0001
10	1.0	0.0001	0.0193	0.0000

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