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Modified Projective Lag Synchronization in Time-Delayed Fractional Order Chaotic Systems

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***Abstract* – Modified projective lag synchronization of fractional order chaotic systems with time-delay is first investigated. It is shown that the slave system is synchronized with the past states of the driver up to a scaling matrix. The synchronization lag time is generally different from the delay time in the coupled systems. According to the stability theorem of linear time-delayed fractional differential systems, a nonlinear feedback controller is proposed for the synchronization of systems with identical and different structures. Numerical simulations are presented to show the effectiveness of the controller.**

***Keywords* – Chaos, Fractional Order, Lag Synchronization, Projective Synchronization, Time-Delay.**

I. INTRODUCTION

Fractional calculus is a generalization of ordinary integration and differentiation to arbitrary (non-integer) order [1]. Although it has been a pure mathematic topic for more than 300 years, its applications to physics and engineering are just a recent subject of interest [2]. Lots of systems in interdisciplinary fields can be described by the fractional differential equations, such as viscoelastic system, dielectric polarization, electrode-electrolyte polarization and financial system. Moreover, the fractional derivatives appear in the control theory of dynamical systems. All of these examples perfectly clarify the importance of consideration and analysis of the fractional order dynamical systems.

Since the inspiring works of Pecora and Carroll [3, 4], chaos synchronization has been a hot subject in the field of nonlinear science due to its wide-scope potential applications in physical systems, biological science, chemical reactor, etc [5, 6]. Many different types of chaos synchronization have been presented, such as complete synchronization [7], projective synchronization [8], lag synchronization [9], anticipated synchronization [10], exponential synchronization [11] and generalized synchronization [12]. Recently, with the introduction of fractional derivative, chaos phenomena and chaotic synchronization of the fractional order dynamical systems become a new active research field owing to its great potential applications especially in secure communication and control processing [13-15]. For example, Chen et al. put forward a sliding mode controller for the synchronization between the integer and fractional order chaotic systems [16]. Si et al. discussed the projective synchronization of fractional order chaotic systems with non-identical orders based on the stability theorem of linear fractional order system [17]. Aghababa considered the finite-time chaos synchronization of fractional chaotic systems based on the fractional Lyapunov stability theorem [18]. And Chen et al. studied the robust synchronization of uncertain fractional order chaotic systems on the basis of linear matrix inequality [19].

The time-delayed characteristics are frequently encountered in the engineering applications and the delayed states are usually presented in the physical, economical and biological systems [20-23]. In 1977, Macky and Glass first found chaos in the time-delayed systems [24]. It has demonstrated that low dimensional chaotic systems do not ensure a sufficient level of security for communications [25]. One way to overcome this problem is by considering high dimensional system with more than one positive Lyapunov exponent. And the time-delayed chaotic systems are just typical of high dimensional chaotic systems [26]. Thus, the time-delayed chaotic systems and its synchronization are intensively studied in recent years [27-29]. For instance, some sufficient conditions were proposed for the complete synchronization of noise-perturbed two bi-directionally coupled integer-order chaotic systems with time-delay [28]. The lag synchronization of coupled time-delayed integer-order chaotic systems and its applications in secure communication were discussed in Ref. [29]. However, most of the existing results focus on the synchronization of integer-order time-delayed chaotic systems. According to the different definitions of fractional derivative from classical integer-order derivative, it is difficult to extend the conclusions on the chaotic synchronization of time-delayed integer-order systems to the cases of time-delayed fractional order systems. In 2011, the complete synchronization for a class of fractional order chaotic systems with varying time-delay is realized based on the Laplace transformation theory [30]. Some other types of synchronization for the non-identical structural time-delayed fractional order chaotic systems are still open problems.

In 1999, Mainieri and Rehacek first presented the projective synchronization [31]. The proportional feature of the projective synchronization extends binary digital to M-nary digital communication for achieving fast communication [32]. Recently, various kinds of projective synchronization for fractional order chaotic systems without time-delay are studied, such as generalized projective synchronization [33], function projective synchronization [34] and modified projective synchronization [35]. In 2014, Wang et al. considered the hybrid projective synchronization for time-delayed fractional order chaotic systems [36]. However, from the viewpoint of engineering applications and channel characteristics, projective synchronization is always practically impossible owing to the signal propagation delays in the environment. For instance, in the telephone communication system, the voice making from the transmitter side at time $t-\tau$ is often heard on the receiver side at time t. Therefore, it is more reasonable that the slave and master systems synchronize with a lag time τ. And the lag synchronization appears as a coincidence of shift-in-time states of interactive chaotic systems in many different areas including lasers [37], complex networks [38], neuron systems [39] and secure communication [40]. In 2011, Chen et al. investigated the projective lag synchronization of fractional order chaotic (hyperchaotic) systems without time-delay based on the stability theorem of linear fractional order systems and the pole placement technique [41]. To the best of our knowledge, there are few results on the lag synchronization of time-delayed fractional order chaotic systems. Motivated by the above discussions, this paper focuses on the modified projective lag synchronization (MPLS) of time-delayed fractional order chaotic systems. The states of the response system synchronize with the past states of the drive system up to a scaling matrix. Complete synchronization, projective synchronization and lag synchronization belong to the MPLS. Both identical and different structural systems can be applied to realize the synchronization.

The remainder of this Letter is organized as follows. In Section II, some preliminaries of fractional calculus are introduced. Then, a nonlinear feedback controller is designed for the MPLS based on the stability theorem of linear time-delayed fractional differential systems in Section III. The numerical simulations in Section IV are applied to manifest the validity and feasibility of the proposed controller. Finally, conclusions are drawn in Section V.

II. Preliminaries

Some basic definitions and properties of fractional calculus are presented as bellow.

A. Definitions of Fractional Calculus

Three fractional derivative definitions, i.e., Grunwald-Letnikov derivative, Riemann-Liouville derivative and Caputo derivative [1] are usually used. The former two definitions are suitable for the pure mathematical problems, while the last one is popular in the real applications. Here we only discuss the Caputo fractional derivative:

$$D^{\beta}v(t)=J^{m-\beta}v^{(m)}(t),\beta>0,$$

where β is the order of fractional derivative, $m=\lceil\beta\rceil$, i.e., m is the first integer which is not less than β, J^{θ} is the θ-order Riemann-Liouville fractional integral operator:

$$J^{\theta}\omega(t)=\frac{1}{\Gamma(\theta)}\int_0^t(t-\tau)^{\theta-1}\omega(\tau)d\tau,\theta>0,$$

where $\Gamma(\cdot)$ is the gamma function. For the function $v(t)$ having m-order continuous derivatives with $t\ge 0$, the Laplace transform of the Caputo fractional derivative is obtained as

$$L\{D^{\beta}v(t);s\}=s^{\beta}V(s)-\sum_{k=0}^{m-1}s^{\beta-k-1}v^{(k)}(0),$$

where $V(s)$ is the Laplace transform of the function $v(t)$, and $v^{(k)}(0)$, $k=0, 1, 2, \cdots, m-1$ are the initial conditions.

B. Stability of a Linear Time-Delayed Fractional Differential System

In Ref. [42], Deng et al. discussed the stability of an n-dimensional linear fractional differential system with multiple time-delays:

$$\begin{aligned}
D^{q_1}z_1(t)&=a_{11}z_1(t-\tau_{11})+a_{12}z_2(t-\tau_{12})+\cdots+a_{1n}z_n(t-\tau_{1n}),\\
D^{q_2}z_2(t)&=a_{21}z_1(t-\tau_{21})+a_{22}z_2(t-\tau_{22})+\cdots+a_{2n}z_n(t-\tau_{2n}),\\
&\vdots\\
D^{q_n}z_n(t)&=a_{n1}z_1(t-\tau_{n1})+a_{n2}z_2(t-\tau_{n2})+\cdots+a_{nn}z_n(t-\tau_{nn}),
\end{aligned}\qquad(1)$$

where $z(t)=(z_1(t), z_2(t), \cdots, z_n(t))^T$ is the state vector, $q_i\in(0, 1)$ is the order of the fractional derivative, $\tau_{ij}>0$ is the time-delay, the initial values $z_i(t)=\phi_i(t)$ are given for $\tau_{ij}=-\tau_{\max}\le t\le 0$, $i, j=1, 2, \cdots, n$, and $A=[a_{ij}]_{n\times n}$ is the coefficient matrix. Applying the Laplace transform to system (1), we obtain

$$\Delta(s)\cdot Z(s)=b(s),$$

where $Z(s)=(Z_1(s), Z_2(s), \cdots, Z_n(s))^T$ is the Laplace transform of state vector $z(t)=(z_1(t), z_2(t), \cdots, z_n(t))^T$, $b(s)=(b_1(s), b_2(s), \cdots, b_n(s))^T$ is the nonlinear part, and the characteristic matrix of system (1) is

$$\Delta(s)=\begin{pmatrix} s^{q_1}-a_{11}e^{-s\tau_{11}} & -a_{12}e^{-s\tau_{12}} & \cdots & -a_{1n}e^{-s\tau_{1n}}\\ -a_{21}e^{-s\tau_{21}} & s^{q_2}-a_{22}e^{-s\tau_{22}} & \cdots & -a_{2n}e^{-s\tau_{2n}}\\ \vdots & \vdots & \ddots & \vdots\\ -a_{n1}e^{-s\tau_{n1}} & -a_{n2}e^{-s\tau_{n2}} & \cdots & s^{q_n}-a_{nn}e^{-s\tau_{nn}}\end{pmatrix}.$$

Theorem 1 [42] If all the roots of the characteristic equation $\det(\Delta(s))=0$ have negative real parts, then the zero solution of system (1) is Lyapunov globally asymptotically stable.

Corollary 1 [42] If $q_1=q_2=\cdots=q_n=\beta\in(0, 1)$, all the eigenvalues λ_i, $i=1, 2, \cdots, n$ of the coefficient matrix A satisfy $|\arg(\lambda_i)|>\beta\pi/2$, and the characteristic equation $\det(\Delta(s))=0$ has no purely imaginary roots for any $\tau_{ij}>0$, $i, j=1, 2, \cdots, n$, then the zero solution of system (1) is Lyapunov globally asymptotically stable.

III. The MPLS of Time-Delayed Fractional Order Chaotic Systems

Consider a time-delayed fractional order drive system as

$$\begin{aligned}D^{\alpha}x(t)&=x(t-\tau)+f(x(t),x(t-\tau)),\\ x(t)&=x(0),t\in[-v,0],\end{aligned}\qquad(2)$$

where $x(t)=(x_1(t), x_2(t), \cdots, x_n(t))^T\in R^n$ is the state vector, $\alpha\in(0, 1)$ is the order of the fractional differential equation, $f:R^{2n}\to R^n$ is a continuous function vector, $\tau>0$ denotes the time-delay and $v>0$ is the initial time. Choose a time-delayed fractional order response system as

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$$D^{\alpha}y(t)=y(t-\tau)+g\left(y(t),y(t-\tau)\right)+u,$$
$$y(t)=y(0),t\in[-\tau,0], \quad (3)$$

where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T\in R^n$ is the state vector, $g:R^{2n}\rightarrow R^n$ is a continuous function vector, and $u = (u_1, u_2, \cdots, u_n)^T$ is a controller which will be designed later.

Definition 1 For the fractional order time-delayed drive system (2) and response system (3), it is said to be the modified projective lag synchronization (MPLS) if there exists a controller u such that

$$\lim_{t\to+\infty}\|e(t)\|=\lim_{t\to+\infty}\|y(t)-\sigma x(t-r)\|=0, \quad (4)$$

where $\sigma=\mathbf{diag}\{\sigma_1, \sigma_2, \cdots,\sigma_n\}$ is a real scaling matrix, $0 < r < v - \tau$ denotes the lag time, $e(t) = y(t)-\sigma x(t-r)$ represents the error state vector between systems (2) and (3) for the MPLS, and $e(t-\tau) = y(t-\tau)-\sigma x(t-\tau-r)$.

Remark 1 If the scaling factors $\sigma_1 = \sigma_2 = \cdots = \sigma_n$, the MPLS is simplified to the lag generalized projective synchronization. Especially, if the scaling matrix σ = I and σ = −I, the MPLS is respectively reduced to the lag synchronization and the anti-phase lag synchronization.

Remark 2 If the lag time r = 0, the MPLS is changed into the modified projective synchronization of fractional order chaotic systems with time-delay. Both complete synchronization and anti-phase synchronization belong to the modified projective synchronization.

Remark 3 If the time-delay τ = 0, the MPLS of time-delayed chaotic systems becomes the projective lag synchronization of fractional order chaotic systems without time-delay [41].

Remark 4 If the time-delay τ = 0 and the lag time r = 0, the MPLS of time-delayed fractional order chaotic systems can be considered as the modified projective synchronization of chaotic systems without time-delay.

Remark 5 Both identical and different structural systems (2) and (3) can be used for the MPLS because the functions $f(x(t), x(t-\tau))$ and $g(y(t), y(t-\tau))$ may be different.

Remark 6 According to the idea of tracking control, $\sigma x(t-r)$ in the error state vector is a reference signal in order to achieve the goal $\lim_{t\to+\infty}\|e(t)\|=0$. Then, the MPLS between systems (2) and (3) belongs to the problem of tracking control, i.e. the output signal y(t) follows the reference signal $\sigma x(t-r)$ ultimately.

With the parameters given above, a nonlinear feedback controller is assumed as

$$u=Ke(t)-g\left(y(t),y(t-\tau)\right)$$
$$+\sigma f\left(x(t-r),x(t-\tau-r)\right), \quad (5)$$

where $K=\mathbf{diag}\{k_1, k_2, \cdots, k_n\}$, $K\in R^{n\times n}$ is a feedback gain matrix which is determined later. Combining (2) and (3) with (5), the fractional order error system is expressed as

$$D^{\alpha}e(t)=y(t-\tau)+g\left(y(t),y(t-\tau)\right)-\sigma x(t-\tau-r)$$
$$-\sigma f\left(x(t-r),x(t-\tau-r)\right)+u \quad (6)$$
$$=Ke(t)+e(t-\tau).$$

Then, the MPLS between the fractional order time-delayed systems (2) and (3) is transformed into the discussion of the asymptotical stability of the zero solution of system (6). According to Corollary 1, a sufficient condition for the MPLS between systems (2) and (3) is gained as follows.

Theorem 2 Given the time-delayed fractional order drive system (2) and response system (3), there exists a feedback gain matrix $K=\mathbf{diag}\{k_1, k_2, \cdots, k_n\}$ in controller (5) such that the MPLS between systems (2) and (3) can be achieved if $k_i < -1/\sin(\alpha\pi/2)$, $i = 1, 2, \cdots, n$.

Proof. Taking the Laplace transform on both sides of the time-delayed fractional order error system (6) gives

$$\Delta(s)\cdot E(s)=s^{\alpha-1}e(0)+e(0)e^{-s\tau}\int_{-\tau}^{0}e^{-st}dt,$$

where $\Delta(s)=s^{\alpha}I-K-e^{-s\tau}I$ is the characteristic matrix, $E(s)$ is the Laplace transform of $e(t)$. The corresponding characteristic equation is

$$\det\left(\Delta(s)\right)$$
$$=\left|s^{\alpha}I-K-e^{-s\tau}I\right| \quad (7)$$
$$=\left(s^{\alpha}-k_1-e^{-s\tau}\right)\left(s^{\alpha}-k_2-e^{-s\tau}\right)\cdots\left(s^{\alpha}-k_n-e^{-s\tau}\right)$$
$$=0.$$

Suppose that $s=\omega i=|\omega|\left(\cos(\pi/2)+i\sin(\pm\pi/2)\right)$ is the root of the following equation

$$s^{\alpha}-k_i-e^{-s\tau}=0, i=1,2,\cdots,n. \quad (8)$$

Then one can get

$$|\omega|^{\alpha}\left(\cos(\alpha\pi/2)+i\sin(\pm\alpha\pi/2)\right)-k_i-\cos(\omega\tau)+i\sin(\omega\tau)=0.$$

Separating the real and imaginary parts, we have

$$|\omega|^{\alpha}\cos(\alpha\pi/2)-k_i=\cos(\omega\tau),$$
$$|\omega|^{\alpha}\sin(\pm\alpha\pi/2)=-\sin(\omega\tau).$$

Hence,

$$|\omega|^{2\alpha}-2k_i\cos(\alpha\pi/2)|\omega|^{\alpha}+k_i^2-1=0. \quad (9)$$

For the given conditions $k_i<-1/\sin(\alpha\pi/2)$, $\alpha\in(0,1)$, we can derive that the discriminant of the roots satisfies

$$\Delta=\left(-2k_i\cos(\alpha\pi/2)\right)^2-4\left(k_i^2-1\right)<0,$$

which means that (9) has no real solutions. Thus, (7) has no purely imaginary roots.

On the other hand, $A = K+I$ is the coefficient matrix of the error system (6). The eigenvalues of the matrix A are $\lambda_i=k_i+1<0$, i = 1, 2, $\cdots$, n due to the given conditions $k_i<-1/\sin(\alpha\pi/2)$, $\alpha\in(0,1)$. Then, all the eigenvalues of A satisfy $|\arg(\lambda_i)|>\pi/2>\alpha\pi/2$, i = 1, 2, $\cdots$, n.

Therefore, when $k_i < -1/\sin(\alpha\pi/2)$, i = 1, 2, $\cdots$, n, the error system (6) meets the requirements of Corollary 1. The zero solution of system (6) is globally asymptotically stable. And the MPLS between the time-delayed fractional order chaotic systems (2) and (3) is realized.

IV. Applications

Numerical examples of the MPLS for time-delayed fractional order chaotic systems with identical and different structures are respectively performed. The effectiveness of the advised synchronization scheme are discussed. And a predictor-corrector scheme [43] is used for the approximate numerical solutions of the time-delayed fractional order differential equations.

A. *MPLS of Time-Delayed Fractional Order Financial Systems*

The MPLS of two identical time-delayed fractional order chaotic financial systems [44] with different initial conditions is considered. The drive system is written as

$$\begin{pmatrix} D^{\alpha}x_1(t) \\ D^{\alpha}x_2(t) \\ D^{\alpha}x_3(t) \end{pmatrix} = \begin{pmatrix} x_{1\tau} \\ x_{2\tau} \\ x_{3\tau} \end{pmatrix} + \begin{pmatrix} x_3(t) - ax_1(t) + x_1(t)x_{2\tau} - x_{1\tau} \\ 1 - bx_2(t) - x_{1\tau}^2 - x_{2\tau} \\ -x_{1\tau} - cx_3(t) - x_{3\tau} \end{pmatrix}, \quad (10)$$

$$x(t) = x(0), t \in [-v, 0],$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the state vector, $x_{i\tau} = x_i(t-\tau)$, $i = 1, 2, 3$, are the simple notations, $\alpha \in (0, 1)$ is the order of the fractional derivative, a, b, c are real positive parameters, $\tau > 0$ denotes the time-delay and $v > 0$ is the initial time. For $\alpha = 0.94$, $v = 0.07$, $\tau = 0.05$, $(a, b, c) = (3, 0.1, 1)$ and $x(0) = (0.5, 2, 1.5)^T$, the chaotic attractor of the time-delayed fractional order financial system (10) is shown in Fig. 1.

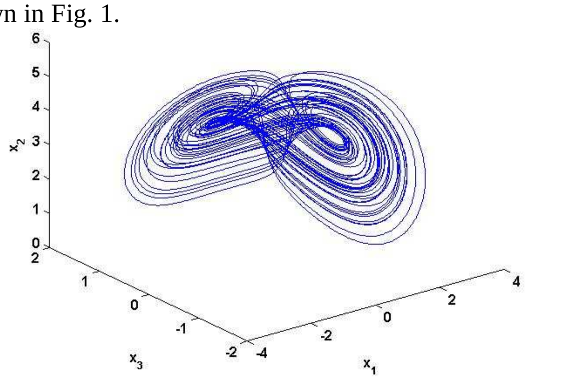


Fig. 1. The chaotic attractor of time-delayed fractional order financial system (10) with $\alpha = 0.94$, $v = 0.07$, $\tau = 0.05$, $(a, b, c) = (3, 0.1, 1)$ and $x(0) = (0.5, 2, 1.5)^T$.

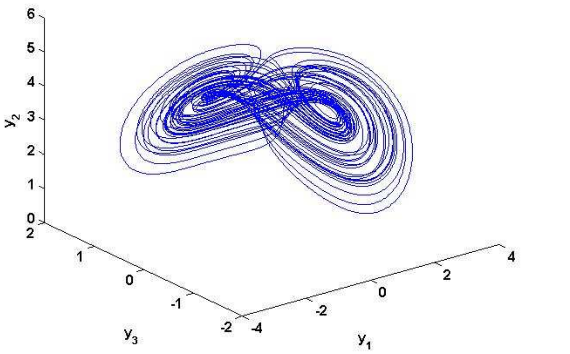


Fig. 2. The chaotic attractor of time-delayed fractional order financial system (11) with $\alpha = 0.94$, $\tau = 0.05$, $(a, b, c) = (3, 0.1, 1)$ and $y(0) = (0.1, 4, 0.5)^T$.

The corresponding response system is described by

$$\begin{pmatrix} D^{\alpha}y_1(t) \\ D^{\alpha}y_2(t) \\ D^{\alpha}y_3(t) \end{pmatrix} = \begin{pmatrix} y_{1\tau} \\ y_{2\tau} \\ y_{3\tau} \end{pmatrix} + \begin{pmatrix} y_3(t) - ay_1(t) + y_1(t)y_{2\tau} - y_{1\tau} \\ 1 - by_2(t) - y_{1\tau}^2 - y_{2\tau} \\ -y_{1\tau} - cy_3(t) - y_{3\tau} \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (11)$$

$$y(t) = y(0), t \in [-\tau, 0],$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ is the state vector, $y_{i\tau} = y_i(t-\tau)$, $i = 1, 2, 3$ are the simple notations, $u = (u_1, u_2, u_3)^{\mathrm{T}}$ is the controller which will be designed later. For $\alpha = 0.94$, $\tau = 0.05$, $(a, b, c) = (3, 0.1, 1)$ and $y(0) = (0.1, 4, 0.5)^T$, the chaotic attractor of the time-delayed fractional order financial system (11) is shown in Fig. 2.

The error state vector between systems (10) and (11) is defined as $e(t) = y(t) - \sigma x(t-r)$, where $e(t) = (e_1(t), e_2(t), e_3(t))^T$, $\sigma = \mathbf{diag}\{\sigma_1, \sigma_2, \sigma_3\}$ is a real scaling matrix, $0 < r < v - \tau$ represents the lag time. Then, $e_i(t) = y_i(t) - \sigma x_i(t-r)$, $i = 1, 2, 3$ and $e(t-\tau) = y(t-\tau) - \sigma x(t-\tau-r)$.

According to the proposed controller (5), the error system is obtained as

$$D^{\alpha}e(t) = Ke(t) + e(t-\tau), \quad (12)$$

where $K = \mathbf{diag}\{k_1, k_2, k_3\}$ is a feedback gain matrix to be determined later. Due to Theorem 2, the MPLS of the time-delayed fractional order chaotic financial systems (10)-(11) is realized with $k_i < -1/\sin(\alpha\pi/2)$, $i = 1, 2, 3$.

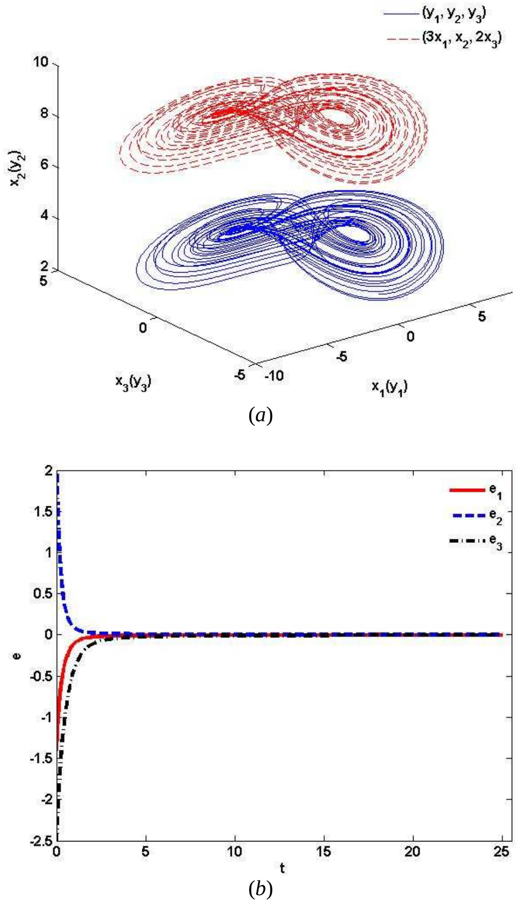
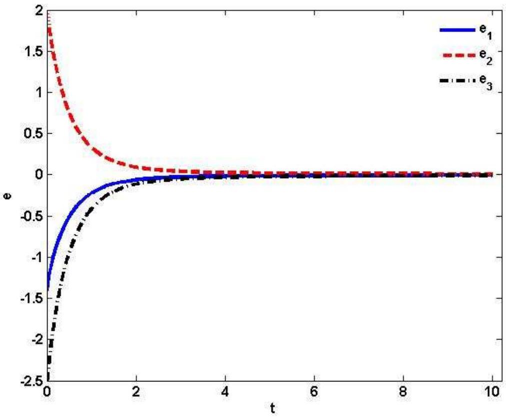
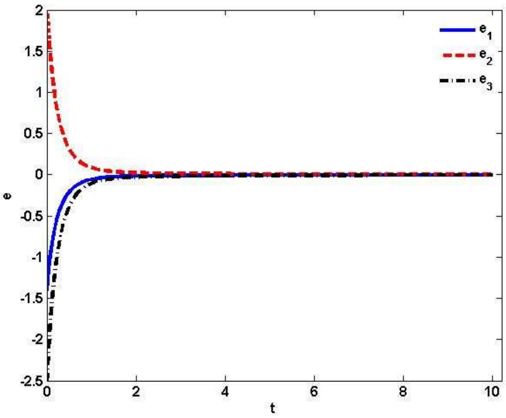


Fig. 3. The MPLS between systems (10) and (11) with $r = 0.01$, $\sigma = \mathbf{diag}\{3, 1, 2\}$, $K = \mathbf{diag}\{-4, -5, -3\}$: (*a*) the system attractors, (*b*) the error state curves.

For example, when $\alpha = 0.94$, $v = 0.07$, $\tau = 0.05$, $(a, b, c) = (3, 0.1, 1)$, $x(0) = (0.5, 2, 1.5)^{\mathrm{T}}$ and $y(0) = (0.1, 4, 0.5)^{\mathrm{T}}$, the drive and response systems (10)-(11) are chaotic. Setting the lag time $r = 0.01$ and the scaling matrix $\sigma = \mathbf{diag}\{3, 1, 2\}$, the MPLS between systems (10) and (11) can be achieved with $K = \mathbf{diag}\{-4, -5, -3\}$ based on Theorem 2. The phase diagrams of systems (10) and (11) are plotted together in Fig. 3(*a*). For clarity of display, the phase diagram of system (10) is moved along the positive direction of the coordinate and the first several points are removed. The corresponding error state curves are shown in Fig. 3(*b*), which indicate that the MPLS between systems (10) and (11) is realized. Moreover, the error state curves are respectively displayed in Figs. 4(*a*)-(*d*) with $k_i = -3$, $k_i = -4$, $k_i = -5$ and $k_i = -7$, $i = 1, 2, 3$, which show that the speed of synchronization can be increased via choosing the smaller values of k_i, $i = 1, 2, 3$.



(*a*)



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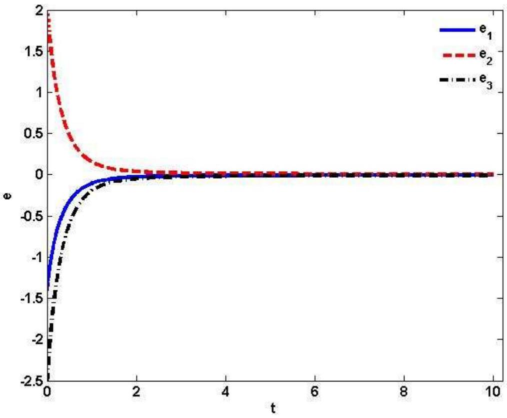
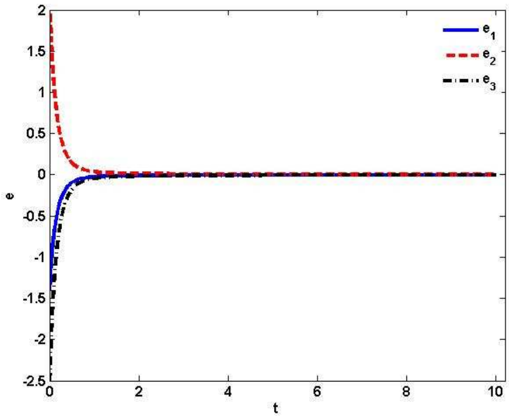


(*b*)



(*d*)

Fig. 4. The error state curves of the MPLS between systems (10) and (11) with $r = 0.01$, $\sigma = \mathbf{diag}\{3, 1, 2\}$: (*a*) $K = \mathbf{diag}\{-3, -3, -3\}$, (*b*) $K = \mathbf{diag}\{-4, -4, -4\}$, (*c*) $K = \mathbf{diag}\{-5, -5, -5\}$, (*d*) $K = \mathbf{diag}\{-7, -7, -7\}$.

B. MPLS Between Time-Delayed Fractional Order Financial and Liu Systems

It is assumed that a time-delayed fractional order Liu system [45] drives the time-delayed fractional order financial system. The drive system is given as

$$\begin{pmatrix} D^{\alpha}x_1(t) \\ D^{\alpha}x_2(t) \\ D^{\alpha}x_3(t) \end{pmatrix} = \begin{pmatrix} x_{1\tau} \\ x_{2\tau} \\ x_{3\tau} \end{pmatrix} + \begin{pmatrix} \theta x_2(t) - \theta x_1(t) - x_{1\tau} \\ \eta x_{1\tau} - x_1(t)x_3(t) - x_{2\tau} \\ -\delta x_{3\tau} - 4x_1^2(t) - x_{3\tau} \end{pmatrix}, \qquad (13)$$

$$x(t) = x(0), t \in [-v, 0],$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ is the state vector, $x_{i\tau} = x_i(t-\tau)$, $i = 1, 2, 3$ are the simple notations, $\alpha \in (0, 1)$ is the order of the fractional derivative, θ, η, δ are the real positive parameters, $\tau > 0$ denotes the time-delay and $v > 0$ is the initial time. For $\alpha = 0.92$, $v = 0.05$, $\tau = 0.01$, $(\theta, \eta, \delta) = (10, 40, 2.5)$ and $x(0) = (2.2, 2.4, 38)^T$, the chaotic attractor of the time-delayed fractional order Liu system (13) is shown in Fig. 5.

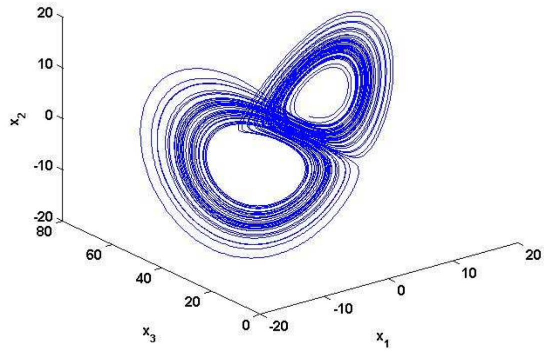


Fig. 5. The chaotic attractor of time-delayed fractional order Liu system (13) with $\alpha = 0.92$, $v = 0.05$, $\tau = 0.01$, $(\theta, \eta, \delta) = (10, 40, 2.5)$ and $x(0) = (2.2, 2.4, 38)^T$.

As mentioned above, the response system can be written as (11). For $\alpha = 0.92$, $\tau = 0.01$, $(a, b, c) = (3, 0.1, 1)$ and $y(0) = (0.5, 2, 1.5)^T$, the chaotic attractor of the time-delayed fractional order financial system (11) is displayed in Fig. 6.

The error state vector between systems (11) and (13) is described by $e(t) = y(t) - \sigma x(t-r)$, where $e(t) = (e_1(t), e_2(t), e_3(t))^T$, $\sigma = \mathbf{diag}\{\sigma_1, \sigma_2, \sigma_3\}$ is a real scaling matrix, $0 < r < v-\tau$ represents the lag time. Then, $e_i(t) = y_i(t) - \sigma x_i(t-r)$, $i = 1, 2, 3$ and $e(t-\tau) = y(t-\tau) - \sigma x(t-\tau-r)$.

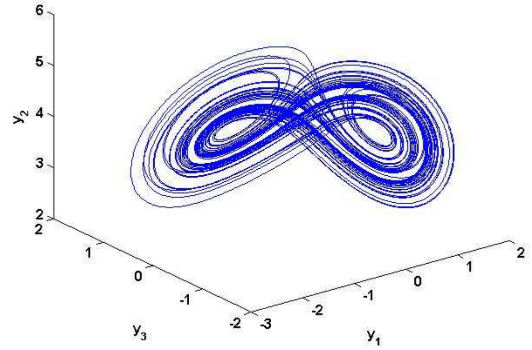


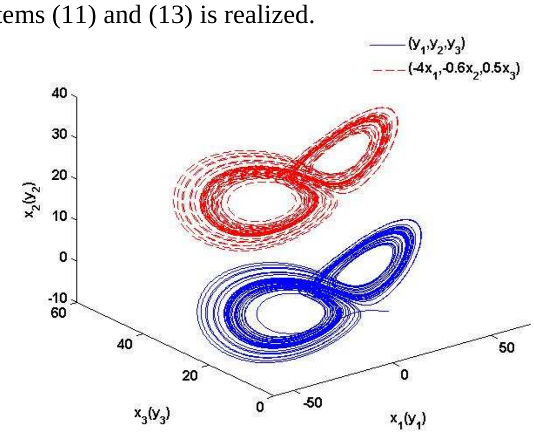
Fig. 6. The chaotic attractor of time-delayed fractional order financial system (11) with $\alpha = 0.92$, $\tau = 0.01$, $(a, b, c) = (3, 0.1, 1)$ and $y(0) = (0.5, 2, 1.5)^T$.

With the proposed controller (5), the error system is obtained as

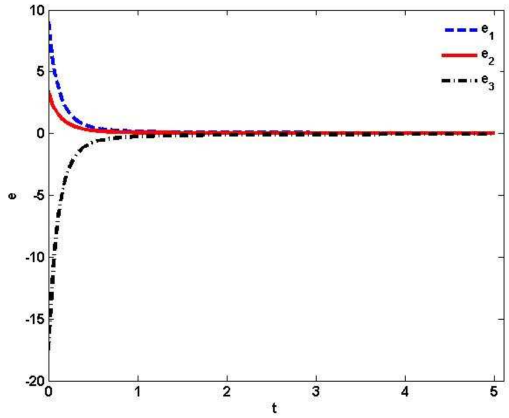
$$D^{\alpha} e(t) = Ke(t) + e(t-\tau), \quad (14)$$

where $K = \mathbf{diag}\{k_1, k_2, k_3\}$ is a feedback gain matrix which will be designed later. Setting $k_i < -1/\sin(\alpha\pi/2)$, $i = 1, 2, 3$, the MPLS between the time-delayed fractional order chaotic financial system (11) and Liu system (13) is realized based on Theorem 2.

For example, when $\alpha = 0.92$, $v = 0.05$, $\tau = 0.01$, $(\theta, \eta, \delta) = (10, 40, 2.5)$, $(a, b, c) = (3, 0.1, 1)$, $x(0) = (2.2, 2.4, 38)^T$ and $y(0) = (0.5, 2, 1.5)^T$, the drive system (13) and the response system (11) are chaotic. Selecting the lag time $r = 0.03$ and the scaling matrix $\sigma = \mathbf{diag}\ \{-4, -0.6, 0.5\}$, the MPLS between systems (11) and (13) can be achieved with $K = \mathbf{diag}\{-8, -7, -9\}$. The phase diagrams of systems (11) and (13) are plotted together in Fig. 7(*a*). For clarity of display, the phase diagram of system (13) is moved along the positive direction of the coordinate and the first several points are removed. The corresponding error state curves are displayed in Fig. 7(*b*), which show that the MPLS between the time-delayed fractional order chaotic systems (11) and (13) is realized.



(*a*)



(*b*)

Fig. 7. The MPLS between systems (11) and (13) with $r = 0.03$, $\sigma = \mathbf{diag}\ \{-4, -0.6, 0.5\}$, $K = \mathbf{diag}\{-8, -7, -9\}$: (*a*) the system attractors, (*b*) the error state curves.

V. Conclusions

The MPLS of time-delayed fractional order chaotic systems is investigated based on the stability theorem of linear fractional order systems with multiple time-delays. The lag synchronization is extended to the time-delayed fractional order chaotic systems. And a nonlinear feedback controller is designed for the synchronization of systems with identical and different structures. Finally, the time-delayed fractional order financial system and Liu system are used for the MPLS to illustrate the effectiveness of the proposed controller. The synchronization speed can be improved by selecting an appropriate matrix K.

In practice, the synchronization is usually destroyed by uncertainties of system parameters. Thus, the synchronization of time-delayed fractional order chaotic systems with unknown parameters will be considered in future.

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