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Further Results on Changing and Unchanging of Total Dominating Color Transversal Number of Graphs

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Abstract **– Total Dominating Color Transversal Set of a graph is a Total Dominating Set of the graph which is also Transversal of Some χ - Partition of the graph. Here χ is the Chromatic number of the graph. Total Dominating Color Transversal number of a graph is the cardinality of a Total Dominating Color Transversal Set which has minimum cardinality among all such sets that the graph admits. In this paper, we prove that a vertex v is a color class of every χ - Partition of G if and only if degree of v is n – 1. Here n is the order of the graph G. We determine a necessary and sufficient conditions under which Total Dominating Color Transversal number decreases, increases or remains same when a vertex, which is a color class of every χ - Partition of G, is removed from the graph. Additionally, we provide related results and sufficient number of examples to justify them.**

Keywords **– Total Domination Number, Total Dominating Color Transversal Number, χ – Partition of a Graph, Transversal.**

I. Introduction

We begin with simple, finite, connected and undirected graph without isolated vertices. We know that proper coloring of vertices of graph G partitions the vertex set V of G into equivalence classes (also called the color classes of G). Using minimum number of colors to properly color all the vertices of G yields χ equivalence classes. Transversal of a χ - Partition of G is a collection of vertices of G that meets all the color classes of the χ – Partition. That is, if T is a subset of V(the vertex set of G) and $\{V_1, V_2, ..., V_\chi\}$ is a χ - Partition of G then T is called a Transversal of this χ - Partition if $T \cap Vi \neq \emptyset$, $\forall$ i $\in \{1, 2,...., \chi\}$.

In [1], we introduced the concept of Total Dominating Color Transversal Set of Graphs. Total Dominating Color Transversal Set of graph G is a Total Dominating Set with the extra property that it is also Transversal of some χ - Partition of G. Total Dominating Color Transversal Set of G with minimum cardinality among all such sets that the graph G admits is called Minimum Total Dominating Color Transversal Set of G and its cardinality, denoted by $\gamma_{tstd}(G)$ or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G. In [8], we determined results regarding changing and unchanging of γ_{tstd} number of a graph by removing a vertex which is not a color class in any χ - Partition of G. In this paper, we compute the operation of removing a vertex which is a color class of every χ - Partition of G and analyze the effect of this operation on γ_{tstd}. As we are dealing with Total Domination theory, we remove only those vertices whose removal does not yield isolated vertices in the resultant sub graph.

II. Definitions

Definition 2.1[4]: (Total Dominating Set) Let G = (V, E) be a graph. Then a subset S of V (the vertex set of G) is said to be a Total Dominating Set of G if for each $v \in V$, v is adjacent to some vertex in S.

Definition 2.2[4]: (Minimum Total Dominating Set/Total Domination number) Let G = (V, E) be a graph. Then a Total Dominating Set S is said to be a minimum Total Dominating Set of G if |S| = minimum {|D|: D is a Total Dominating Set of G}. Here S is called γ_t –set and its cardinality, denoted by $\gamma_t(G)$ or just by γ_t, is called the Total Domination number of G.

Definition 2.3[1]: (χ -partition of a graph) Proper coloring of vertices of a graph G, by using minimum number of colors, yields minimum number of independent subsets of vertex set of G called equivalence classes (also called color classes of G). Such a partition of a vertex set of G is called a χ - Partition of the graph G.

Definition 2.4[1]: (Transversal of a χ - Partition of a graph) Let G = (V, E) be a graph with χ – Partition $\{V_1, V_2, ..., V_\chi\}$. Then a set $S \subset V$ is called a Transversal of this χ – Partition if $S \cap V_i \neq \emptyset$, $\forall$ i $\in \{1, 2, 3, ..., \chi\}$.

Definition 2.5[1]: (Total Dominating Color Transversal Set) Let G = (V, E) be a graph. Then a Total Dominating Set $S \subset V$ is called a Total Dominating Color Transversal Set of G if it is Transversal of at least one χ - Partition of G.

Definition 2.6[1]: (Minimum Total Dominating Color Transversal Set/Total Dominating Color Transversal number) Let G = (V, E) be a graph. Then $S \subset V$ is called a Minimum Total Dominating Color Transversal Set of G if |S| = minimum {|D|: D is a Total Dominating Color Transversal Set of G}. Here S is called γ_{tstd} –Set and its cardinality, denoted by $\gamma_{tstd}(G)$ or just by γ_{tstd}, is called the Total Dominating Color Transversal number of G.

Definition 2.7: (Isolated Vertex) Let G = (V, E) be a graph. Then a vertex v of G is isolated vertex if its degree is 0.

Definition 2.8: (Isolate vertex of a set) Let G = (V, E) be a graph and S be a subset of V. Then a vertex v in S is called isolate of S if v is not adjacent to any vertex in S.

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III. Main Results

We first mention the following basic results and theorems that will prove vital later on.

Result 3.1 [7]: Let G be a graph and H be a sub graph of G. Then χ (H) ≤ χ (G).

Result 3.2: Let G= (V, E) be a graph and v ∈ V. If χ (G \ {v}) < χ (G) then χ (G \ {v}) + 1 = χ (G). (G \ {v} be a sub graph of G obtained by removing vertex v from G)

Theorem 3.3 [11]: Let G = (V, E) be a graph and v ∈ V. Then χ (G \ {v}) = χ (G) if and only if {v} is not a color class of any χ - Partition of G.

Proof: Assume χ (G \ {v}) = χ (G). Suppose {v} is a color class of some χ – Partition $\prod = \{V_1, V_2, ..., V_\chi\}$ of G with $\{v\} = V_1$. Clearly$\{V_2, ..., V_\chi\}$ is a χ – 1 partition of G \ {v}, which is contradiction. So {v} is not a color class of any χ - Partition of G.

Conversely assume that {v} is not a color class of any χ - Partition of G. Suppose χ (G \ {v}) < χ (G). Let $\{V_1, V_2, ..., V_{\chi-1}\}$be a χ - 1 Partition of G \ {v}. Then $\{\{v\}, V_1, V_2, ..., V_{\chi-1}\}$ is a χ – Partition of G, which is contradiction. Hence χ (G \ {v}) = χ (G).

Theorem 3.4 [10]: Let G = (V, E) be a graph and S ⊂ V is an independent set. If S is not maximal independent set then it is a not a color class of some χ - Partition of G.

Now we present one important theorem about a vertex which is color class of every χ - Partition of G.

Theorem 3.5: Let G = (V, E) be a graph of order n and v ∈ V. Then v is a color class of every χ - Partition of G if and only if deg (v) = n – 1.

Proof: Assume {v} is a color class of every χ - Partition of G. Suppose deg (v) ≠ n – 1. Then v is not adjacent to some vertex u ∈ V. So {u, v} is independent set in G. Therefore {v} is not maximal independent set of G. Hence theorem 3.4, {v} is not a color class of some χ - Partition of G, which is contradiction. Hence deg (v) = n – 1.

Conversely assume deg (v) = n – 1. If {v} is not a color class of some χ - Partition of G then there exists a vertex with which v is not adjacent, which is again contradiction. Hence {v} is a color class of every χ - Partition of G.

Before going further we introduce some notations.

Notations: Let G = (V, E) be a graph.

(1) V^i = {v ∈ V/ G \ {v} has an isolated vertex}

(2) V^0_{tstd} = {v ∈ V/ Υ_{tstd} (G \ {v}) = Υ_{tstd} (G)}

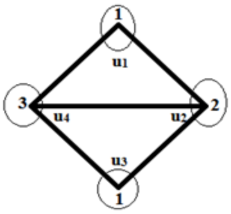
(3) V^-_{tstd} = {v ∈ V/ Υ_{tstd} (G \ {v}) < Υ_{tstd} (G)}

(4) V^+_{tstd} = {v ∈ V/ Υ_{tstd} (G \ {v}) > Υ_{tstd} (G)}

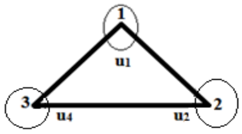
Clearly all the above sets are mutually disjoint and their union is the vertex set V of G.

First of all let us see some examples which shows that this number Υ_{tstd} (G) may remain same, increase or decrease by removing a vertex of degree n – 1.(n is the order of the graph)

Example 3.6: For the below given graph G, Υ_{tstd} (G \ {u_3}) = Υ_{tstd} (G)



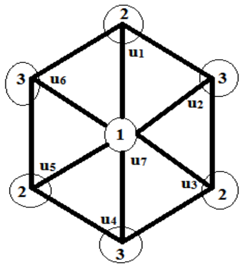
G
Fig. 1



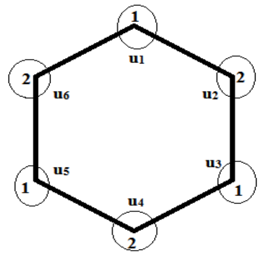
G \ {u_3}
Fig. 2

Υ_{tstd} (G) = 3 and Υ_{tstd} (G \ {u_3}) = 3.

Example 3.7: For the below given graph G, Υ_{tstd}(G \{u_7})>Υ_{tstd}(G).



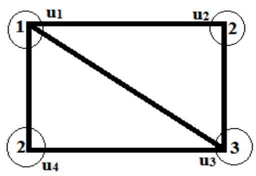
G
Fig. 3



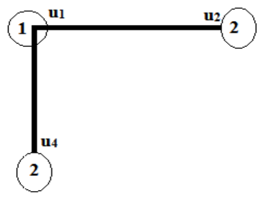
G \ {u_7}
Fig. 4

Υ_{tstd} (G) = 3 and Υ_{tstd} (G \ {u_7}) = 4.

Example 3.8: For the given graph G, Υ_{tstd} (G \ {u_3}) < Υ_{tstd} (G).



G
Fig. 5



G \ {u_3}
Fig. 6

Υ_{tstd} (G) = 3 and Υ_{tstd} (G \ {u_3}) = 2

Theorem 3.9: Let G = (V, E) be a graph and v ∈ V with v ∉ V^i and {v} is a color class of every χ - Partition of G. If v ∈ V^-_{tstd} then Υ_{tstd} (G \ {v}) + 1 = Υ_{tstd} (G).

Proof: Assume v ∈ V^-_{tstd}.

As {v} is a color class of every χ - Partition of G, χ (G \ {v}) + 1 = χ (G). Let D be a Υ_{tstd} - Set of G \ {v} for the χ - 1 - Partition $\{V_1, V_2, ..., V_{\chi-1}\}$ of G \ {v}. Note that deg (v) = n -1 . So D is a total dominating set of G. Also D ∪ {v} is a Total Dominating Color Transversal Set of G for the χ - Partition $\{\{v\}, V_1, V_2, ..., V_{\chi-1}\}$ of G . So Υ_{tstd} (G) ≤ Υ_{tstd} (G \ {v}) + 1. Hence by Υ_{tstd} (G \ {v}) < Υ_{tstd} (G) we have Υ_{tstd} (G \ {v}) + 1 = Υ_{tstd} (G).

Theorem 3.10 [1]: If γ_t(G) = 2 then γ_{tstd}(G) = χ (G).

Theorem 3.11 [9]: Let G be a graph. Then Υ_{tstd} (G) = χ (G) > Υ_t (G) if and only if following two conditions hold:

(1) There exists a Total Dominating Set D of G and a χ – Partition $\{V_1, V_2, ..., V_\chi\}$ of G such that $|D \cap V_i| = 1$, ∀ i = 1, 2, ... , χ.

(2) No Υ_t - Set of G is a transversal of any χ – Partition of G.

Proof: We know that $\Upsilon_{tstd}(G) \geq \chi(G)$.

Suppose (1) is not true. Then $\Upsilon_{tstd}(G) > \chi(G)$, which is contradiction to $\Upsilon_{tstd}(G) = \chi(G)$. Hence (1) is true.

Suppose (2) is not true. Then there exists a Υ_t - Set of G which is a transversal of some χ – Partition of G. Therefore $\Upsilon_{tstd}(G) = \Upsilon_t(G)$ which is contradiction to $\Upsilon_{tstd}(G) > \Upsilon_t(G)$. Hence (2) is true.

Conversely assume (1) and (2). (1) implies that $\Upsilon_{tstd}(G) \leq \chi(G)$ which implies that $\Upsilon_{tstd}(G) = \chi(G)$ and (2) implies that $\Upsilon_{tstd}(G) > \Upsilon_t(G)$. So $\chi(G) = \Upsilon_{tstd}(G) > \Upsilon_t(G)$.

Now we are at a stage where we can introduce our first theorem about necessary and sufficient condition under which Υ_{tstd} number for a graph decrease when a vertex, that is a color class of every χ – Partition of G, is removed.

Theorem 3.12: Let G = (V, E) be a graph and v ∈ V with $v \notin V^i$ and {v} is a color class of every χ - Partition of G. Then $v \in V^-_{tstd}$ if and only if there exists a Total Dominating Set, say D, of G \ {v}and a $\chi - 1$ - Partition { $W_1, W_2, \ldots, W_{\chi-1}$} of G \ {v} such that $|D \cap W_i| = 1, \forall i = 1, 2, \ldots, \chi - 1$.

Proof: Assume $v \in V^-_{tstd}$.

As {v} is a color class of every χ - Partition of G, $\Upsilon_t(G) = 2$. By theorem 3.10, $\Upsilon_{tstd}(G) = \chi(G)$. Then by theorem 3.11, there exists a Total Dominating Set S of G and a χ – Partition $\{V_1, V_2, \ldots, V_\chi\}$ of G such that $|S \cap V_i| = 1, \forall i = 1, 2, \ldots, \chi$.

Let {v} = V_1(without loss of generality). Now as $v \in V^-_{tstd}$, $\Upsilon_{tstd}(G \setminus \{v\}) + 1 = \Upsilon_{tstd}(G)$.(by theorem 3.9). So $\Upsilon_{tstd}(G \setminus \{v\}) + 1 = \chi(G)$. Therefore $\chi(G \setminus \{v\}) = \chi(G) - 1 = \Upsilon_{tstd}(G \setminus \{v\})$. Then by theorem 3.11, there exists a Total Dominating Set D of G \ {v} and a $\chi - 1$ - Partition { $W_1, W_2, \ldots, W_{\chi-1}$} of G \ {v} such that $|D \cap W_i| = 1, \forall i = 1, 2, \ldots, \chi - 1$.

Conversely assume that there exists a Total Dominating Set, say D, of G \ {v} and a $\chi - 1$ - Partition { $W_1, W_2, \ldots, W_{\chi-1}$}of G \ {v} such that $|D \cap W_i| = 1, \forall i = 1, 2, \ldots, \chi - 1$. Then D becomes Total Dominating Color Transversal Set of G \ {v}. So $\Upsilon_{tstd}(G \setminus \{v\})) \leq \chi(G \setminus \{v\})$. But $\chi(G \setminus \{v\})) \leq \Upsilon_{tstd}(G \setminus \{v\})$. Therefore $\Upsilon_{tstd}(G \setminus \{v\}) = \chi(G \setminus \{v\}) < \chi(G) = \Upsilon_{tstd}(G)$.

Hence $v \in V^-_{tstd}$.

Theorem 3.14: Let G = (V, E) be a graph and v ∈ V with $v \notin V^i$ and {v} is a color class of every χ - Partition of G. Then $v \in V^0_{tstd}$ if and only if following two conditions hold:

(1) There exists a Total Dominating Set, say D, of G \ {v}and a $\chi - 1$ - Partition { $V_1, V_2, \ldots, V_{\chi-1}$} of G \ {v} such that $|D \cap V_i| = 1, \forall i = 1, 2, \ldots, j - 1, j + 1, \ldots, \chi - 1$ and $|D \cap V_j| = 2$, for some $j \neq i$.
(2) $\Upsilon_{tstd}(G \setminus \{v\}) > \chi(G \setminus \{v\})$.

Proof: As {v} is a color class of every χ - Partition of G, $\Upsilon_t(G) = 2$. By theorem 3.10, $\Upsilon_{tstd}(G) = \chi(G)$. Also $\chi(G \setminus \{v\}) = \chi(G) - 1$.

Assume $v \in V^0_{tstd}$. So $\chi(G) = \Upsilon_{tstd}(G) = \Upsilon_{tstd}(G \setminus \{v\})$. So there exists a Total Dominating Set, say D, of G \ {v}) and a χ - 1 - Partition { $V_1, V_2, \ldots, V_{\chi-1}$}of G \ {v}such that $|D \cap V_i| = 1, \forall i = 1, 2, \ldots, j - 1, j + 1, \ldots, \chi - 1$ and $|D \cap V_j| = 2$, for some $j \neq i$. So (1) holds. Also as $\Upsilon_{tstd}(G \setminus \{v\}) = \Upsilon_{tstd}(G) = \chi(G) > \chi(G \setminus \{v\})$. So (2) holds.

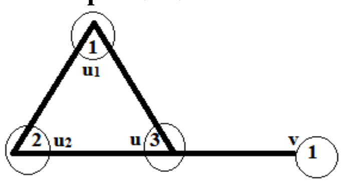
Conversely assume (1) and (2). Then $\chi(G) - 1 = \chi(G \setminus \{v\}) < \Upsilon_{tstd}(G \setminus \{v\}) \leq |D| = \chi(G)$. So $\Upsilon_{tstd}(G \setminus \{v\}) = \chi(G) = \Upsilon_{tstd}(G)$. Hence $v \in V^0_{tstd}$.

Corollary 3.17: Let G = (V, E) be a graph and v ∈ V with $v \notin V^i$ and {v} is a color class of every χ - Partition of G. Then $v \in V^0_{tstd}$ if and only if there exists a Υ_{tstd} - Set, say D, of G \ {v} and a χ - 1 partition { $V_1, V_2, \ldots, V_{\chi-1}$}of G \ {v} such that $|D \cap V_i| = 1, \forall i = 1, 2, \ldots, j - 1, j + 1, \ldots, \chi - 1$ and $|D \cap V_j| = 2$, for some $j \neq i$.

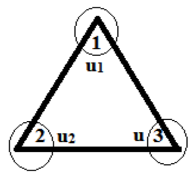
Theorem 3.18: Let G = (V, E) be a graph and v, u ∈ V with $v \notin V^i$. If {v} is not a color class of any χ - Partition of G and {u} is a color class of every χ - Partition of G then $v \in V^0_{tstd}$ with $\Upsilon_{tstd}(G \setminus \{v\}) = \Upsilon_{tstd}(G) = \chi(G)$.

Proof: As {u} is a color class of every χ - Partition of G, $\Upsilon_t(G) = 2$. By theorem 3.10, $\Upsilon_{tstd}(G) = \chi(G)$. Also $\chi(G \setminus \{v\}) = \chi(G)$, by theorem 3.3. Note that $\Upsilon_t(G \setminus \{v\}) = 2$. So $\Upsilon_{tstd}(G \setminus \{v\}) = \chi(G \setminus \{v\}) = \chi(G) = \Upsilon_{tstd}(G)$ and hence $v \in V^0_{tstd}$.

Example 3.19:



G
Fig. 7



G \ {v}
Fig. 8

$\Upsilon_{tstd}(G) = \Upsilon_{tstd}(G \setminus \{v\}) = 3 = \chi(G)$.

Remark 3.20: Converse of above theorem 3.19 is not true in general. Take an example of graph C_4 in which no vertex is a color class any χ - Partition of G but still $\Upsilon_{tstd}(G) = \Upsilon_{tstd}(G \setminus \{v\}) = \chi(G) = 2$.

Example 3.21: Consider G = C_4. Then $\Upsilon_{tstd}(G \setminus \{v\}) = \Upsilon_{tstd}(G), \forall v \in V$.

Example 3.22: Consider G = K_n, ($n \geq 3$) (K_n is a complete graph with n vertices). Then $\Upsilon_{tstd}(G \setminus \{v\}) < \Upsilon_{tstd}(G), \forall v \in V$.

Example 3.23: Consider G = $W_{13} = C_{13} \vee K_1$ (The wheel graph). Then $\Upsilon_{tstd}(G) = 3$ and $\Upsilon_{tstd}(G \setminus \{v\}) = 7$, where v is a hub vertex. Hence Υ_{tstd} number may increases by more than 1 by removal of a vertex.

IV. Conclusion

In this paper, we have removed only those vertices whose degree is n – 1. (where n is the order of the graph). We have noticed that Total Dominating Color Transversal number may increase, decrease or remain same. We proved that this number decreases by at most one. Also we have seen an example of a graph in which removal of a vertex may increase this number by more than one.

Removal of a arbitrary vertex, without any restriction on its degree, can also be discussed and hence can be considered as further scope of study in this area.

REFERENCES

[1] D. K. Thakkar and A. B. Kothiya, Total Dominating Color Transversal number of Graphs, Annals of Pure and Applied Mathematics, Vol. 11(2), 2016, 39 – 44.

[2] R. L. J. Manoharan, Dominating colour transversals in graphs, Bharathidasan University, September, 2009.

[3] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, Fundamentals of domination in graphs, Marcel Dekker, NewYork, 1998.

[4] Michael A. Henning and Andres Yeo, Total Domination in Graphs, Springer, 2013.

[5] Sandi Klavzar, "Coloring Graph Products – A survey", Elsevier Discrete Mathematics 155 (1996), 135- 145.

[6] R. Balakrishnan and K. Ranganathan, "A Textbook of Graph Theory, Springer", New York, 2000.

[7] Goksen BACAK, " Vertex Color of a Graph", Master of Science Thesis, ˙IZM˙IR, December, 2004.

[8] D.K.Thakkar and A.B.Kothiya, Changing and unchanging of Total Dominating Color Transversal number of Graphs, accepted for publication in International Journal of scientific and innovative Mathematical Research.

[9] Further results on Total Dominating Color Transversal number of Graphs, Communicated for publication in Advances and Applications in Discrete Mathematics.

[10] D.K.Thakkar and A.B.Kothiya, Relation Between Dominating and Total Dominating Color Transversal number of Graphs, International Journal of Mathematics and its Applications, Vol.4(1-D), 2016, 111 - 118.

[11] D.K.Thakkar and A.B.Kothiya, Relation Between Total Dominating Color Transversal number and Chromatic number of a Graph, Communicated for publication in AKCE International Journal of Graphs and Combinatorics.

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