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Fixed Points of Expansion Mappings in Fuzzy Menger Spaces with CLRs Property

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***Abstract* – In this paper, we prove existence of unique common fixed point for two pairs of weakly compatible mappings using common limit in range property in fuzzy menger spaces. Our result generalizes several known results of Menger spaces as well as metric spaces.**

***Keywords* – Fuzzy Menger Space, Non-Surjective Mappings, Weakly Compatible Mappings, Expansion Mappings, Property (E.A.) CLRs Property.**

I. Introduction

The concept of metric space in which notion of distance appears was introduced by Ferchet [3], where for any two points in the space, there is defined a positive number called the distance between the two points. However, in practice we find very often that this association of a single number for each pair is strictly speaking an over-idealization. Therefore, Menger [8] introduced the notion of a probabilistic metric space in 1942 (briefly PM space) as a generalization of metric space. Since then the theory of probabilistic metric spaces has developed in many directions [12]. The idea of Menger was to use distribution functions instead of non-negative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis.

Banach contraction principle [1] is an important tool in the field of fixed point theory. Due to its simplicity and usefulness, it became a very popular tool in solving existence problems in pure and applied science such as biology, medicine, physics and computer science. Probabilistic contraction were first defined and studied by Sehgal [12]. Banach contraction principle [1] also yields a fixed point theorems for a diametrically opposite class of mapping, which was expansion mapping. The study of metrical fixed point theorem for expansion mapping is initiated by wang et al. [17]. There after pant et al. [10] studied fixed point theorems for expansion mappings in framework of probabilistic metric spaces and so many authors (see [2], [4], [12], [14], [16]) worked along this direction. Shrivastav et al. [15] given the definition of fuzzy probabilistic metric space and proved fixed point theorem in such spaces.

II. Preliminaries

Definition 2.1 A fuzzy probabilistic metric space (FPM space) is an ordered pair (X, F_α) consisting of a nonempty set X and a mapping F_α from XxX into the collection of all fuzzy distribution function $F_\alpha \in R$ for all $\alpha \in [0,1]$. For x, y ∈ X we denote the fuzzy distribution function $F_\alpha(x,y)$ by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)} at\ u\ in\ R$.

The function $F_{\alpha(x,y)}$ for all $\alpha \in [0,1]$ assumed to satisfy the following conditions.

(a) $F_{\alpha(x,y)}(u) = 1\ \forall\ u > 0\ iff\ x = y$,

(b) $F_{\alpha(x,y)}(0) = 0\ \forall\ x, y$ in X,

(c) $F_{\alpha(x,y)} = F_{\alpha(y,x)}\ \forall\ x, y$ in X,

(d) If $F_{\alpha(x,y)}(u) = 1$ and $F_{\alpha(y,z)}(v) = 1 \Rightarrow F_{\alpha(x,z)}(u+v) = 1\ \forall\ x, y, z \in X$ and $u, v > 0$

Definition 2.2 A commutative associative and non-decreasing mapping t: [0, 1] x [0,1] →[0,1] is a t-norm if and only if t(a,1) = a ∀ a ∈ [0,1], $t(0,0) = 0$ and $t(c,d) \geq t(a,b)$ for $c \geq a,\ d \geq b$.

Definition 2.3 A Fuzzy Menger space is a triplet (X, F_α, t) where (X, F_α) is a FPM-space, t is a t-norm and generalized triangle inequality.

$$F_{\alpha(x,z)}(u+v) \geq t(F_{\alpha(x,z)}(u), F_{\alpha(y,z)}(v))$$

Holds for all x, y, z in X. u, v > 0 and $\alpha \in [0,1]$.

The concept of neighborhoods in fuzzy Menger space is introduced as.

Definition 2.4 Let (X, F_α, t) be a Fuzzy Menger space. if $x \in X,\ \varepsilon > 0\ and\ \lambda \in (0,1)$ then (ε, λ) neighborhood of x called $U_x(\varepsilon, \lambda)$ is deffined by

$$U_x(\varepsilon, \lambda) = \{y \in X, : F_{\alpha(x,y)}(\varepsilon) > (1 - \lambda)\}$$

An (ε, λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon, \lambda): x \in X, \varepsilon > 0, \alpha \in [0,1] and\ \lambda \in (0,1)\}$ } of neighborhood.

Remark: If t is continuous, then Fuzzy Menger space (X, F_α, t) is a Housdroff space in (ϵ, λ)-topology.

Let (X, F_α, t) be a complete fuzzy Menger space and A ⊂ X. Then A is called a bounded set if

$$\lim_{u\to\infty} \inf F_{\alpha(x,y)}(u) = 1$$

Definition 2.5 A sequence $\{X_n\}$ in (X, F_α, t) is said to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer N=N(ε, λ) such that $x_n \in U_x(\varepsilon, \lambda)\ \forall\ n \geq N$ or equivalently $F_\alpha(x_n, x; \varepsilon) > 1 - \lambda$ for all n ≥N and $\alpha \in [0,1]$.

Definition 2.6 A sequence $\{x_n\}$ in (X, F_α, t) is said to be Cauchy sequence if for every ε >0 and $\lambda > 0, there$ exists an integer N=N(ε, λ) such that for all α∈[0,1] $F_\alpha(x_n, x_m; \varepsilon) > 1 - \lambda$ for all n, m ≥ N.

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Definition 2.7 A fuzzy Menger Space (X, F_α, t) with continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all α ∈ [0, 1].

Following lemmas are selected from [8] and [12] respectively in fuzzy Menger space.

Lemma 2.1. Let $\{x_n\}$ be a sequence in a Fuzzy Menger space (X, F_α, t) with continuous t-norm * and t * t ≥ t. if there exists a constant k ∈ (0, 1) such that $F_{\alpha(x_n, x_{n+1})}(kt) \geq F_{\alpha(x_{n-1}, x_n)}(t)$ for all t > 0, α ∈ [0,1] and n =1,2………

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2 Let (X, F_α, t) be a Fuzzy Menger space. If there exists k ∈ (0,1) such that

$F_{\alpha(x,y)}(kt) \geq F_{\alpha(x,y)}(t)$ for all x, y ∈ X and for all α ∈ [0,1] and t > 0 and then x=y.

Definition 2.8 [5] A pair (A, S) of self mapping of a non-empty set X is said to be weakly compatible (or Coincidentally commuting) if they commute at their coincidence points, that is if Az = Sz some z ∈ X then ASz = SAz.

Two compatible self maps are weakly compatible, but the converse is not true. Therefore the concept of weak compatibility is more general than that of compatibility.

Definition 2.9. A pair (A, S) of self mapping of Fuzzy Menger space (X, F_α, t) is said to satisfy the property (E. A), if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z ∈ X.

Definition 2.10. A pair (A, S) of self mappings of Fuzzy Menger space (X, F_α, t) is said to satisfy the common limit in range of S (in short CLRs property), if there exists a sequences $\{x_n\}$ such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = Su$ for some u ∈ X.

III. Main Result

Now we prove our main result

Theorem 3.1. Let A, B, S and T be four self mappings of a fuzzy Menger space (X, F_α, t). Suppose that

(3.1) (A, S) satisfies CLRs property or (B, T) satisfies CLR_T property;

(3.2) A(X) ⊆ T(X), B(X) ⊆ S(X);

(3.3) (A, S) and (B, T) are weak compatible;

(3.4) One of the range of the mapping A, B, S or T is a closed subspace of X;

(3.5) There exist a constant k >1 such that

$F_{\alpha(Ax,By)}(kt) \leq F_{\alpha(Sx,Ty)}(t)$

for all x, y ∈ X for all α ∈ [0, 1] and t >0.

Then A, B, S and T have a unique common fixed point in X.

Proof: - If the pair (A, S) satisfies the CLRs property, then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, where z ∈ S(X).

Therefore, there exists a point u in X such that z= Su

Since A(X) ⊆ T(X), there exists a sequence $\{y_n\}$ in X such that

$Ax_n = Ty_n$. Hence $\lim_{n\to\infty} Ty_n = z$

Thus, we have

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = z$

Now we are required to show that $\lim_{n\to\infty} By_n = z$. Putting $x = x_n$ and $y = y_n$ in (3.5) we get

$F_{\alpha(Ax_n,By_n)}(kt) \leq F_{\alpha(Sx_n,Ty_n)}(t)$

Let $\lim_{n\to\infty} By_n = l \neq z$ then taking limit as n→∞, we have

$$F_{\alpha(z,l)}(kt) \leq F_{\alpha(z,z)}(t) = 1,$$

for all t > 0, α ∈ [0, 1] and k > 1.

By Lemma 2.2, we have z=l, thus $\lim_{n\to\infty} By_n = z$.

Hence

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z = Su$$

for some u ∈ X.

Now we claim that Au = Su. For this purpose putting x=u and y = y_n in (3.5), we have

$F_{\alpha(Au,By_n)}(kt) \leq F_{\alpha(Su,Ty_n)}(t)$,

taking limit as n→∞, we gets

$F_{\alpha(Au,Su)}(kt) \leq F_{\alpha(Su,Tu)}(t) = 1$ for all t > 0, α ∈ [0,1] and k >1

By Lemma 2.2, we have

Au = Su. Hence Au = Su =z.

Since A(X) ⊆ T(X) there exists a point v in X such that Au= Tv= z.

Next we claim that Tv = Bu, putting x=u and y = v in (3.5), we have

$$F_{\alpha(Au,Bv)}(kt) \leq F_{\alpha(Su,Tv)}(t),$$

or

$$F_{\alpha(Au,Bv)}(kt) \leq F_{\alpha(z,z)}(t),$$

for all t> 0, α ∈ [0, 1] and k>1. By Lemma 2.2, we have Au =Bv. Hence Au= Tv = Bv = z.

Since the pairs (A,S) and (B,T) are weakly compatible also Au=Su and Bv=Tv, we have

Az = ASu = SAu= Sz and Bz=BTv = TBv = Tz

Now, we prove that z is common fixed point of A, B, S and T. Putting x=z and y=v in (3.5), we have

$$F_{\alpha(Az,Bv)}(kt) \leq F_{\alpha(Sz,Tv)}(t)$$

or

$$F_{\alpha(Az,z)}(kt) \leq F_{\alpha(Az,z)}(t)$$

By Lemma 2.2, we have Az = z. Hence Az=Sz=z. Similarly we can prove that Bz = Tz = z. Thus z is common fixed point of A, B, S and T.

For uniqueness, suppose z ≠ w is another fixed point of A, B, S and T.

Then using (3.5) , we have

$$F_{\alpha(Az,Bw)}(kt) \leq F_{\alpha(Sz,Tw)}(t)$$

or

$$F_{\alpha(z,w)}(kt) \leq F_{\alpha(z,w)}(t).$$

Appealing to Lemma 2.2, if follows that z=w. This completes the proof.

The proof is similar if we assume that one of the subspace B(X), S(X) or T(X) is closed.

Remark 3.6. The conclusion of theorem 3.1 remains true if we replace (3.5) by one of the following: for all k>1, x, y >0, α ∈ [0,1] and t > 0

(3.6)

$$F_{\alpha(Ax,By)}(kt) \leq \min\{F_{\alpha(Sx,Ty)}(t),\ F_{\alpha(Ax,Sx)}(t),\ F_{\alpha(By,Ty)}(t)\}$$

(3.7) $(F_{\alpha(Ax,By)}(kt))^2 \leq F_{\alpha(Ax,Sx)}(t),\ F_{\alpha(By,Ty)}(t)\}$

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By setting A = B and S = T in Theorem 3.1, we can obtain a natural result for a pair of self mappings.

Corollary 3.1. Let A and S ne two self mapping of a Fuzzy Menger space(X, F_α, t). Suppose that

(3.10) S(X)⊆A(X);

(3.11) (A, S) satisfies the CLR_S property;

(3.12) one the range of the mapping A or S is a closed subspace of X

(3.13) There exists a constant k>1 such that

$F_{\alpha(Ax,Ay)}(kt) \leq F_{\alpha(Sx,Sy)}(t)$

for all x, y ∈ X, α ∈ [0,1]and t>0.

Then A and S have a unique common fixed point in X.

IV. Conclusion

Theorem 3.1 is a generalization of some results in the sense it proved for non-surjective mapping under weak compatibility which is more general than compatibility and without any requirement of completeness of the whole space and continuity of involved mappings.

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