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A Priori Estimates in the Production Planning Task

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***Abstract* – A basic plans group of production planning was marked out and investigated. The boundary values of the problem variables were identified. On this basis a priori estimates of the optimal values of the objective functions of its direct and complementary problems were obtained. The geometric locus of the angular points set was defined for every problem. The additional opportunities for post optimal analysis are shown. Using a suitable surface area the estimates can be adjusted. An algorithm for determining a given accuracy of the optimal value of the objective function is offered, a description of the corresponding computer program is given. An ellipsoid containing the range of possible solutions of a problem is defined. Examples are given.**

***Keywords* – Basic Plan, Complementary Problem, Evaluation, Objective Function, Tangent Point.**

Application of mathematical methods for optimizing production planning was investigated for the first time by George B. Dantzig and L.W. Kantorowitsch [1], [2]. After that methods of linear programming [4], [7]as well as parametric programming [5], [8] were elaborated. It is of interest to consider a pair of mutually complementary linear programming problems.

Let us examine a problem of production planning [3] and a complementary one.

Direct problem:

$f(x) = \sum_{j=1}^{n} c_j x_j = max$ (1)

$\sum_{j=1}^{n} a_{ij} x_j \le b_i$ i= $1, \dots, m$ (2)

$x_j \ge 0$ j= 1,…,n (3)

Pursuant to a famous economic interpretation, the aim is to maximize profit of some production under the given prices, resources volumes and standard expenditures of the latter for production of a unit of every type. It is also supposed

$c_j > 0, b_i > 0, a_{ij} \ge 0 \ (i = 1, \dots, m; j = 1, \dots, n)$

Suppose all the basic plans of a problem are nondegenerate.

Complementary problem:

$f(x) = \sum_{j=1}^{n} c_j x_j = min$ (4)

$\sum_{j=1}^{n} a_{ij} x_j \ge b_i$ i= $1, \dots, m$ (5)

$x_j \ge 0 \ i = 1, \dots, n$ (6)

The objective of the given essay is a priori identification of variables´ boundary values in problems (1)-(3) и (4)-(6), and also estimate of the optimal values of the corresponding objective functions. At the same time this information is undoubtedly of theoretical and practical interest. Particularly, there appears a possibility to compare, under the post optimal analysis, the values of variables in the optimal solution with their boundary values.

Let us examine hyperplanes:

$\sum_{j=1}^{n} a_{ij} x_j = b_i$ i=1,…,m (7)

Reduce them to the following form:

$\sum_{j=1}^{n} \frac{x_j}{(b_i/a_{ij})} = 1,$ $a_{ij} \ne 0;$ i=1,…,m

The values $\frac{b_i}{a_{ij}}$(j=1,…,n ; i=1,…,m) are intervals, cut with hyperplanes on the coordinate axes (7). Let us define:

$d_j = min_{i=1}^{i=m} \frac{b_i}{a_{ij}}$, $e_j = max_{i=1}^{i=m} \frac{b_i}{a_{ij}}$, j=1,…,n.

Geometrically d_j is the lowest one of the intercept points of the coordinate axis x_j with the hyperplanes (7).

Points D_1 (d_1 , 0,…,0), D_2 (0, d_2 ,0,…,0), …, D_n (0,…,0, d_n) are the polyhedron´s apexes of the conditions(2)-(3), i.e. basic plans of the direct problem. They belong to the hyperplane:

$\sum_{j=1}^{j=n} \frac{x_j}{d_j} = 1.$ (8)

The rest of the polyhedron´s apexes belong to the half-space, that does not contain point O (0, …,0).

Geometrically e_j is the highest one of the intercept points of the coordinate axis x_j with the hyperplanes (7). Analogously, points E_1 (e_1,0,…,0), E_2 (0, e_2, 0,…,0), … , E_n (0,…,0,e_n) are the apexes of the polyhedral set, given by the conditions (5)–(6), i.e. by support plans of the complementary problem. They belong to the hyperplane:

$$\sum_{j=1}^{j=n} \frac{x_j}{e_j} = 1.$$

The rest of the apexes of the polyhedral set belong to the half-space, that contains point O(0, ….,0).

Let us define:

f_d-optimal value of the direct problem´s objective function;

f_k- optimal value of the complementary problem´s objective function.

We have:

$$f_d \ge max_{j=1}^{j=n} f(D_j) = max_{j=1}^{j=n} c_j d_j = max_{j=1}^{j=n} c_j min_{i=1}^{i=m} \frac{b_i}{a_{ij}}; \quad (9)$$

$$f_k \le min_{j=1}^{j=n} f(E_j) = min_{j=1}^{j=n} c_j e_j = min_{j=1}^{j=n} c_j max_{i=1}^{i=m} \frac{b_i}{a_{ij}}; \quad (10)$$

$$max_{j=1}^{j=n} c_j min_{i=1}^{i=m} \frac{b_i}{a_{ij}} \le f_d \le f_k \le min_{j=1}^{j=n} c_j max_{i=1}^{i=m} \frac{b_i}{a_{ij}}. \quad (11)$$

Suppose

$0 \le x_j \le d_j$ j=1,…,n (12)

Let us examine a problem of linear programming (1)-(3). Suppose m ≤ n. If in the optimal solution l (0<l≤ m) resources have been used up completely, then n-l variables will apply the boundary values, consequently, l variables will apply the intermediate values. The other way round: if l variables in the optimal solution have applied the intermediate values, then n-l variables will apply the boundary values, consequently, l resources will be used up

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completely. If l = m = n all resources are fully utilized, all the variables x_j take intermediate values. Let K - the only common point of the solution sets of direct and complementary problems; G_k - a set of support hyperplanes passing through the point K.

Suppose that the objective function of the direct problem is parallel to one of them. The value of the objective function at the point K is optimal for both applications. According to the second duality theorem, it will also be optimal for the corresponding dual problems. Example ([6]).

Let us examine a problem of linear programming:

G(x) = $18x_1 + 9x_2 + 17x_3 + 20x_4$ = max

$1{,}2x_1 + 0{,}5x_2 + 0{,}8x_3 + 1{,}6x_4 \le 500$;

$1{,}6x_1 + 0{,}5x_2 + 1{,}0x_3 + 2{,}0x_4 \le 550$;

$x_1, x_2, x_3, x_4 \ge 0$.

The point D (0,600,250,0) gives the optimal solution.

Maximum value of the objective function G(D) = 18×0 + 9×600 + 17×250 +20×0 = 9650.

Equations of restraint hyperplanes:

$1{,}2x_1 + 0{,}5x_2 + 0{,}8x_3 + 1{,}6x_4 = 500$;=>

$1{,}6x_1 + 0{,}5x_2 + 1{,}0x_3 + 2{,}0x_4 = 550$.

$$\frac{x_1}{500/1{,}2} + \frac{x_2}{500/0{,}5} + \frac{x_3}{500/0{,}8} + \frac{x_4}{500/1{,}6} = 1;\Rightarrow$$

$$\frac{x_1}{550/1{,}6} + \frac{x_2}{550/0{,}5} + \frac{x_3}{550/1{,}0} + \frac{x_4}{550/2{,}0} = 1$$

$$\frac{x_1}{416{,}7} + \frac{x_2}{1000} + \frac{x_3}{625} + \frac{x_4}{312{,}5} = 1;$$

$$\frac{x_1}{343{,}75} + \frac{x_2}{1100} + \frac{x_3}{550} + \frac{x_4}{275} = 1.$$

Let us fill in the table 1 of objective function´s values in points $D_i(x_1,x_2,x_3,x_4)$:

Table 1. Values of the objective functions

i	d_i	$D_i(x_1,x_2,x_3,x_4)$	G(D_i)
1	d_1 = min (416,7; 343,75) = 343,75	D_1(343,75; 0; 0; 0)	G(D_1) = 18×343,75 = 6187,5
2	d_2 = min (1000; 1100) = 1000	D_2(0; 1000; 0; 0)	G(D_2) = 9×1000 = 9000
3	d_3 = min (625; 550) = 550	D_3(0; 0; 550; 0)	**G(D_3) = 17×550 = 9350**
4	d_4 = min (312,5; 275) = 275	D_4(0; 0; 0; 275)	G(D_4) = 20×275 = 5500

Boundary values of the variables:

$0 \le x_1 \le 343{,}75$;

$0 \le x_2 \le 1000$;

$0 \le x_3 \le 550$;

$0 \le x_4 \le 275$.

Equation of the hyperplane going through the points D_1-D_4:

$$\frac{x_1}{343{,}75} + \frac{x_2}{1000} + \frac{x_3}{550} + \frac{x_4}{275} = 1.$$

The rest of the apexes of the polyhedron belong to the half-space, that does not contain the point O(0, …, 0).

Below there is assessment of the objective function´s value:

$max_{i=1}^{i=4} G(D_i)$ = 9350; $\frac{9650-9350}{9650}$ ×100% = 3,1%.

Let us plug the optimal solution into the constraints:

1,2×0 + 0,5×600 +0,8×250 +1,6×0 = 500;

1,6×0 + 0,5×600 + 1,0×250 + 2,0×0 = 550,

i.e. both resources are used up completely (l=m=2).

Consequently, the two variables apply the boundary values:

$x_1 = 0$; $x_4 = 0$.

The other two variables apply the intermediate values:

0< x_2 = 600 < 1000; 0< x_3 = 250 < 550.

These values are remote enough from the boundary intervals and close to the middle of corresponding ones**.** The estimates of the objective function´s optimal solution, given above, can be adjusted by using a suitable surface area. Let us examine the tangential hyperplane to the surface area parallel to the objective function. The value of the objective function at the tangency point is taken for the estimate of the objective function´s optimal value. Plugging of the tangency point´s coordinates into the constraints lets identify a type of estimate (upper or lower). For such a surface area the ellipsoid can be taken:

$$\sum_{i=1}^{i=n} \frac{x_i^2}{d_i^2} = 1. \qquad (13)$$

Therewith the tangency points will be not negative.

Also it would be possible to use surface areas of the family:

$$\sum_{i=1}^{i=n} \left(\frac{x_i}{d_i}\right)^{1+\alpha} = 1,\ 0<\alpha\le 1. \qquad (14)$$

At α close to 0 the surface area approaches the hyperplane (8).

At α=1 the surface area coincides with the ellipsoid (13).

When using unlimited surface areas non-negativity of all the coordinates of the tangency point is not guaranteed. When there are diverse estimates an interval for the optimal value of the objectivefunction can be identified. An analogical analysis can be conducted for a complementary problem. The corresponding estimates can be used for the direct problem and vice versa.

Let us illustrate the said above with an example from [6] (the objective function is changed).

Consider the problem of linear programming:

G(x) = $x_1 + x_2$ = max

$x_1 + 4x_2 \le 22$,

$2x_1 + 3x_2 \le 19$,

$3x_1 + 2x_2 \le 21$,

$x_1, x_2 \ge 0$.

The point D (5I3) gives the optimal solution of the problem.

The value of the objective function at the point G(D) = 3+5=8.

We have:

d_1=min($\frac{22}{1}$;$\frac{19}{2}$;$\frac{21}{3}$)=7, d_2=min($\frac{22}{4}$;$\frac{19}{3}$;$\frac{21}{2}$)=5,5.

D_1 (7I0), D_2(0I5,5).

Equation of the corresponding ellipsis:

$\frac{x_1^2}{7^2}+\frac{x_2^2}{5,5^2}=1.$

Further

$\frac{\frac{2x_1}{7^2}}{1}=\frac{\frac{2x_2}{5,5^2}}{1}$; $x_1=\frac{7^2}{5,5^2}x_2$; $\frac{1}{7^2}(\frac{7^2}{5,5^2}x_2)^2+\frac{x_2^2}{5,5^2}=1$; $x_2^2\frac{7^2+5,5^2}{5,5^4}=1$;

$x_2=+\frac{5,5^2}{\sqrt{7^2+5,5^2}}=3,40$; $x_1=\frac{7^2}{5,5^2}3,40=5,51$;

$G(x_1,x_2)=5,51+3,40=8,91$.

The tangential line to the ellipsis that is parallel to the objective function touches the ellipsis at the point F (5,51I3,40). The point F does not belong to the domain of the problem. Estimate of the objective function on the top equals 8,91. We receive estimate at the bottom using an in equation (9):

$\max((G(D_1);G(D_2))=\max(7+0;0+5,5)=7$.

Thus, the optimal value of the objective function belongs to the interval (7;8,91).

These estimates can be adjusted. Further we use a parabola with the vertex at the point D_2, that crosses the axes x_1 at the point D_1.

$x_2(x_1)=ax_1^2+5,5$; $0=a\cdot 7^2+5,5$; $a=-0,112$;$x_2(x_1)=-0,112x_1^2+5,5$

$x_2'(x_1)=-2\cdot 0,112x_1=-0,224x_1$; $-1=-0,224x_{10}$;$x_{10}=4,46$;$x_{20}(4,46)=-0,112\cdot 4,46^2+5,5=3,27$

$F_1(4,46I3,27)$; $G(F_1)=4,46+3,27=7,73$

The point F_1 belongs to the solution domain of the problem, so that the estimate received is the estimate at the bottom.

Now let us use the parabola with the vertex at the point D_1 that crosses the axes x_2 at the point D_2 .

$x_1(x_2)=b\cdot x_2^2+7$; $0=b\cdot 5,5^2+7$; $b=-0,23$;$x_1(x_2)=-0,23\cdot x_2^2+7nn$

$x_1'(x_2)=-2\cdot 0,23x_2=-0,46x_2$; $-1=-0,46x_{20}$; $x_{20}=2,173$;$x_{10}(2,173)=-0,23\cdot 2,173^2+7=5,91$

$F_2(5,91I2,173)$; $G(F_2)=G(5,91I2,173)=5.91+2,173=8,08$

The point F_2does not belong to the solution domain of the problem. The estimate received is the estimate on the top.

Thus, the optimal solution of the objective function belongs to the interval (7,73;8,08).

We can certify a noticeable decrease of the interval´s length that contains the optimal value of the objective function.

In general case (see. (14)) for calculating the coordinates of a tangency point and the corresponding objective function´s values the following formulas can be used:

$$x_i=\frac{d_i c_i^{1/\alpha}}{(\sum_{k=1}^{k=n} c_k^{1+1/\alpha})^{1/(1+\alpha)}},\quad i=1,\dots,n \qquad (15)$$

$$G(x_1,\dots,x_n)=\frac{\sum_{i=1}^{i=n} d_i c_i^{1+1/\alpha}}{(\sum_{k=1}^{k=n} c_k^{1+1/\alpha})^{1/(1+\alpha)}}. \qquad (16)$$

Formulas derivation:

1) $\frac{(1+\alpha)(\frac{x_1}{d_1})^\alpha}{c_1}=\dots=\frac{(1+\alpha)(\frac{x_n}{d_n})^\alpha}{c_n}$.
2) $(\frac{x_i}{d_i})^\alpha=\frac{c_i}{c_n}(\frac{x_n}{d_n})^\alpha$, i=1,…,n-1.
3) $\frac{x_i}{d_i}=(\frac{c_i}{c_n})^{\frac{1}{\alpha}}\frac{x_n}{d_n}$, i=1,…,n-1.
4) $(\frac{x_i}{d_i})^{1+\alpha}=(\frac{c_i}{c_n})^{1+\frac{1}{\alpha}}(\frac{x_n}{d_n})^{1+\alpha}$, i=1,…,n-1.
5) $\sum_{i=1}^{i=n-1}(\frac{x_i}{d_i})^{1+\alpha}+(\frac{x_n}{d_n})^{1+\alpha}=1$.
6) $\sum_{i=1}^{i=n-1}(\frac{c_i}{c_n})^{1+\frac{1}{\alpha}}(\frac{x_n}{d_n})^{1+\alpha}+(\frac{x_n}{d_n})^{1+\alpha}=1$.
7) $\frac{1}{c_n^{1+\frac{1}{\alpha}}}(\frac{x_n}{d_n})^{1+\alpha}(\sum_{i=1}^{i=n-1}c_i^{1+\frac{1}{\alpha}}+c_n^{1+\frac{1}{\alpha}})=1$.
8) $(\frac{x_n}{d_n})^{1+\alpha}=\frac{c_n^{1+\frac{1}{\alpha}}}{\sum_{i=1}^{i=n}c_i^{1+\frac{1}{\alpha}}}=\frac{c_n^{\frac{1+\alpha}{\alpha}}}{\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}}}$.
9) $\frac{x_n}{d_n}=\frac{c_n^{\frac{1}{\alpha}}}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}}$
10) $\frac{x_i}{d_i}=(\frac{c_i}{c_n})^{\frac{1}{\alpha}}\frac{c_n^{\frac{1}{\alpha}}}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}}$, i=1,…,n-1.
11) $x_i=\frac{c_i^{\frac{1}{\alpha}}d_i}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}}$, i=1,…,n.
12) $\sum_{i=1}^{i=n}c_i x_i=\sum_{i=1}^{i=n}\frac{c_i^{\frac{1+\alpha}{\alpha}}d_i}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}}=\frac{\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}}d_i}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}}$.
13) In particular,at α=1:
$\sum_{i=1}^{i=n}c_i x_i=\frac{\sum_{i=1}^{i=n}c_i^2 d_i}{\sqrt{\sum_{i=1}^{i=n}c_i^2}}$.

Sequential changes in the value of the parameter α (increases if the tangency point belongs to the domain of the problem, and decreases in the opposite case) will let determine the optimal value of the objective function (16) with a given accuracy.

Description of the corresponding computer program:

1. $\alpha:=\alpha_0$ (initial value); $G^*:=0$
2. Calculation of x_j, G.
3. ε −IG - G^*I ≥ 0? Yes – 8. No – 4.
4. b_i-$\sum_{j=1}^{n}a_{ij}\,x_j$≥0? Yes – 5. No – 7.
5. Increase of α.
6. G^*:=G. Transition to 2.
7. Decrease of α. Transition to 6.
8. Derivation of x_j, G.
9. The end.

Similar arguments can be made for complementary problem (4) - (6).

For this purpose, a family of surface areas of the following form can be used:

$$\sum_{i=1}^{i=n}\left(\frac{I\,x_i-e_i\,I}{e_i}\right)^{1+\alpha}=1,\qquad \alpha\geq 0. \qquad (17)$$

Further:

$$x_i-e_i=-\frac{c_i^{\frac{1}{\alpha}}d_i}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}},\qquad i=1,\dots,n. \qquad (18)$$

$$x_i=e_i-\frac{c_i^{\frac{1}{\alpha}}d_i}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}},\qquad i=1,\dots,n. \qquad (19)$$

$$\sum_{i=1}^{i=n}c_i\,x_i=\sum_{i=1}^{i=n}c_i(e_i-\frac{c_i^{\frac{1}{\alpha}}d_i}{(\sum_{i=1}^{i=n}c_i^{\frac{1+\alpha}{\alpha}})^{\frac{1}{1+\alpha}}}). \qquad (20)$$

From (12) follows:

$\frac{x_i}{d_i}\leq 1$; $(\frac{x_i}{d_i})^2\leq 1$; $\sum_{i=1}^{i=n}(\frac{x_i}{d_i})^2\leq n$; $\sum_{i=1}^{i=n}(\frac{x_i}{d_i'})^2\leq 1$, $d_i'=\sqrt{n}d_i$. (21)

Thus, the domain of the problem (1)-(3) can be found within the ellipsoid:

$$\sum_{i=1}^{i=n}(\frac{x_i}{d_i'})^2=1,\ d_i'=\sqrt{n}d_i. \qquad (22)$$

The point $D(d_1,\dots,d_n)$ belongs to this ellipsoid.

The estimate of the optimal value of the objective function on the top (see (16) at α=1)**:**

$$f_d \le \frac{\sum_{i=1}^{i=n} d'_i c_i^2}{\sqrt{\sum_{i=1}^{i=n} c_i^2}} = \frac{\sqrt{n}\sum_{i=1}^{i=n} \mathrm{d_i} c_i^2}{\sqrt{\sum_{i=1}^{i=n} c_i^2}}. \qquad (23)$$

Let us examine the family of ellipsoids:

$$\sum_{i=1}^{i=n}\left(\frac{x_i}{k\sqrt{n}d_i}\right)^2=1, \qquad 0<k\le 1. \qquad (24)$$

Like above, we examine a tangency hyperplane to the surface area of the family that is parallel to the objective function. The value of the objective function at the tangency point is taken for the estimate of the objective function´s optimal value. Sequential changes in the value of the parameter k (increases if the tangency point belongs to the domain of the problem, and decreases in the opposite case) will let determine the optimal value of the objective function with a given accuracy.

Calculation formulas for x_i and $G(x_1,\dots,x_n)$ are derived from (15) and (16) at α=1 and plugging $k\sqrt{n}d_i$ instead of d_i.

$$x_i=\frac{c_i d_i}{\sqrt{\sum_{k=1}^{k=n} c_k^2}}=\frac{k\sqrt{n}c_i d_i}{\sqrt{\sum_{k=1}^{k=n} c_k^2}}, \quad i=1,\dots,n \qquad (25)$$

$$G(x_1,\dots,x_n)=\sum_{i=1}^{i=n} c_i x_i = \sum_{i=1}^{i=n} c_i \frac{k\sqrt{n}c_i d_i}{\sqrt{\sum_{k=1}^{k=n} c_k^2}}=\frac{k\sqrt{n}}{\sqrt{\sum_{k=1}^{k=n} c_k^2}}\sum_{i=1}^{i=n} c_i^2 d_i . \qquad (26)$$

In the description of the corresponding computer program given above the parameter α should be substituted with the parameter k. As an initial value k_0=1 can be taken.

Thus, there is a possibility of a priori estimation of the parameters of the optimal solution of production planning problems. Approximate determination of the values of these parameters can be performed using an interactive procedure.

REFERENCES

[1] George B. Dantzig: *Linear Programming and Extensions.* Rand Corp., Santa Monica 1959).

[2] L. W. Kantorowitsch: *Mathematical Methods of Organizing and Planning Production.* In: *Management Science.* Vol. 6, No. 4, Jul. 1960, S. 366–422.

[3] A. Koop, H. Moock. Lineare Optiemierung. Eine anwendungsorientierte Einfürung in Operations Research. Springer-Verlag Berlin Heidelberg, 2008.

[4] P. Stingl. Operations Research. Fachbuchverlag Leipzig im Carl Hauser Verlag, München,2002.

[5] W. Dinkelsbach. Sensivitäts Analysen und Parametrische Programmierung. Springer Verlag, Berlin, 1969.

[6] B. Luderer, U. Würker. Einstieg in die Wirtschaftsmathematik. B.G.Teubner, Stuttgart, Leipzig, 2000.

[7] Alexander Schrijver: Theory of Linear and Integer Programming. John Wiley and Sons, 1998, ISBN 0-471-98232-6.

[8] Saaty, T.L., S.I. Gass: The parametric objective funktion 1, Oper. Res. 2 (1954) 316-319.

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