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On Non-Homogeneous Cubic Diophantine Equation $5x^2 + 5y^2 - 9xy = 23z^3$

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Abstract **– Four different methods of the non-zero non-negative solutions of non- homogeneous cubic Diophantine equation $5x^2 + 5y^2 - 9xy = 23z^3$ are exposed. Introducing the linear transformation $x = u + v$, $y = u - v$, $u \neq v \neq 0$ in $5x^2 + 5y^2 - 9xy = 23z^3$, it reduces to $u^2 + 19v^2 = 23z^3$. We are solved the above equation through various choices and the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are observed. The following notations are used: $t_{n,m}$ = Polygonal number of rank n with sides m- G_n= Gnomonic number of rank n – P_n^m = Pyramidal number of sides n with rank m 2010 Mathematics subject classification: 11D25**

Keywords **– The Method of Factorization, Integer Solutions, Linear Transformation, Relations and Special Numbers.**

I. Introduction

The number theory is queen of Mathematics. In particular, the Diophantine equations have a blend of attracted interesting problems. For an extensive review of variety of problems, one may refer to [1-21]. Integer solutions of cubic Diophantine Equation $x^2 + y^2 - xy = 103z^3$ has appeared in Jayakumar. P, Meena, J [16, 18]. In 2016, Jayakumar. P, Pandian. P, Venkatraman .V has published a paper [21] in finding the integer solutions of the cubic Diophantine equation: $4x^2 + 4y^2 - 7xy = 19z^3$ Inspired by these, we are observed in this work another interesting four different methods of the non-zero integral solutions of non- homogeneous cubic Diophantine equation $5x^2 + 5y^2 - 9xy = 23z^3$. Further, some elegant properties among the special numbers and the solutions are observed.

II. Description of Method

Consider the cubic Diophantine equation

$5x^2+5y^2-9xy = 23z^3$ (1)

We take the linear transformations

$x = u + v,\ y = u - v,\ u \neq v \neq 0$ (2)

Using (2) in (1), it gives to $u^2 + 19v^2 = 23z^3$ (3)

If we take $z = z(a, b) = a^2+19b^2 = (a + i\sqrt{19}b)(a - \sqrt{19}\,b)$ (4)

where a and b non-zero distinct integers, then we solve (1) through dissimilar method of solutions of (1) which are furnished below.

1 *Method: I*

We can write 23 as $23 = (2 + i\sqrt{19})(2- i\sqrt{19})$

(5) Using (4) and (5) in (3) and applying factorization process, this gives us

$u + i\sqrt{19}\,v)(u - i\sqrt{19}v) = (2+i\sqrt{19})(2-i\sqrt{19})$
$(a+ i\sqrt{19}b)^3(a- i\sqrt{19}\,b)^3$

This gives us

$(u + i\sqrt{19}v) = (2+ i\sqrt{19})(a + i\sqrt{19}\,b)^3$ (6)

$(u - i\sqrt{19}\,v) = (2 - i\sqrt{19})(a - i\sqrt{19}\,b)^3$ (7)

It gives us

$u = u(a, b) = 2a^3 - 57a^2b - 114ab^2 + 361b^3$

$v = v(a, b) = a^3 + 6a^2b - 57ab^2 - 38b^3$

In true of (2), the values of *x*, *y* are given by

$x = x(a, b) = 3a^3 - 51a^2b - 171ab^2 + 323b^3$ (8)

$y = y(a, b) = a^3 - 63a^2b - 57ab^2 + 399b^3$ (9)

Hence (4), (8) and (9) gives us two parametric the non-zero different integral values of (1).

Observations

1. z (9a, a) is a perfect square
2. $\frac{1}{11}$ [y(a, a) –x(a, a)]is a perfect square
3. $x(a, 1) - 6P_a^5 + 54P_a + G_{55a} \equiv 0 \pmod 7$.
4. $x(1,b) - 446P_b^5 + 494t_{4,b} \equiv 0 \pmod 3$
5. $y(a, 1) - 2P_a^5 + 64P_a + G_{13b} \equiv 0 \pmod 7$.
6. $y(1, b) - 798P_b^5 - 456P_b - 393b = 1$

Each of the following is a nasty number

7. $\frac{6}{5}$ z (a, a), $\frac{3}{10}$ z (9a, a), 2x (1, 0)

2.2 *Method: II*

Take 23 as $23 = \frac{1}{196}(67 + i\sqrt{19})(67 - i\sqrt{19})$ (10)

Using (4) and (10) in (3) and applying the process of factorization, this gives us

$(u + i\sqrt{19}\,v)(u - i\sqrt{19}\,v) = \frac{1}{14}[(67+i\sqrt{19})$
$(67 - i\sqrt{19})(a + i\sqrt{19}\,b)^3(a - i\sqrt{19}b)^3$

Equating the positive and negative factors, we get

$(u + i\sqrt{19}\,v) = \frac{1}{14}[(67 + i\sqrt{19})(a + i\sqrt{19}b)^3]$ (11)

$(u - i\sqrt{19}v) = \frac{1}{14}[(67 - i\sqrt{19})(a - i\sqrt{19}\,b)^3]$ (12)

It gives us

$u = u(a, b) = \frac{1}{14}[67a^3 - 57a^2b - 3819ab^2 + 361b^3]$ (13)

$v = v(a, b) = \frac{1}{14}[a^3 + 201a^2b - 157ab^2 - 1273b^3]$ (14)

In sight of (2), the values of x, y are found to be

$x = x(a, b) = \frac{1}{14}(68a^3 + 114a^2b - 3976ab^2 - 912b^3)$ (15)

$y = y(a, b) = \frac{1}{14}(66a^3 - 250a^2b - 3662ab^2 + 1634b^3)$ (16)

Since our intension is to find integer solutions, taking a as 7a and b as 7b in (4), (13) and (14), the related parametric integer values of (1) are found as

$x = x(a, b) = 1666a^3 + 3528a^2b - 97412ab^2 - 223442b^3$ (17)

$y = y(a, b) = 1617a^3 - 6321a^2b - 89719ab^2 + 40033b^3]$ (18)

$z = z(a, b) = 49a^2 + 931b^2$ (19)

Hence (17), (18) and (19) gives us two parametric the non-zero different integral values of (1).

Observations:

1. $\frac{6}{245} z(a, a)$ is a nasty number
2. $x(a, 1) - 3332P_a^5 - 1862P_a + G_{4963a} \equiv 0 \pmod 5$
3. $x(1,b) + 44688P_b^5 - 12603P_b + G_{4535b} \equiv 0 \pmod 5$
4. $y(a, 1) - 3234P_a^5 + 7938P_a + G_{40889a} \equiv 0 \pmod 2$
5. $y(1, b) - 8066P_b^5 + 129752P_b - G_{61714b} \equiv 0 \pmod 3$

2.3 *Method: III*

Write (3) as $u^2 + 3v^2 = 103z^3 * 1$ (20)

Write 1 as $1 = \frac{1}{100}(9 + i\sqrt{19})(9 + i\sqrt{19})$ (21)

Using (4), (5) and (16) in (15) and applying the process of factorization, we are found as

$(u + i\sqrt{19}v)(u - i\sqrt{19}v) = 1/4 (9 + i\sqrt{19})(9 - i\sqrt{19})$
$(2 + i\sqrt{19})(2 - i\sqrt{19})(a + i\sqrt{19}b)^3(a - i\sqrt{19}b)^3$

Equating the positive and negative factors, we get

$(u + i\sqrt{19}v) = \frac{1}{10}(9 + i\sqrt{19})(2 + i\sqrt{19})(a + i\sqrt{19}b)^3$ (22)

$(u - i\sqrt{19}v) = \frac{1}{10}(9 - i\sqrt{19})(2 - i\sqrt{19})(a - i\sqrt{19}b)^3$ (23)

It gives

$u = u(a, b) = \frac{1}{10}[-a^3 - 627a^2b + 57ab^2 + 3971b^3]$

$v = v(a, b) = \frac{1}{10}[11a^3 - 3a^2b - 627ab^2 + 19b^3]$

In view of (2), the values of x, y are given by

$x = x(a, b) = \frac{1}{10}[10a^3 - 630a^2b - 570ab^2 + 3990b^3]$ (24)

$y = y(a, b) = \frac{1}{10}[-12ab^3 - 624a^2b + 684ab^2 + 3952b^3]$ (25)

As our intension is to find integer solutions, taking a as 5a and b as 5b in (4), (24) and (25), the related parametric integer values of (1) are found as

$x = x(a, b) = 125a^3 - 7875a^2b - 7125ab^2 + 49875b^3$
$y = y(a, b) = -15ab^3 - 7800a^2b + 8550ab^2 + 49400b^3$
$z = z(a, b) = 25a^2 + 475b^2$

Hence the above three equations give us two parametric the non-zero different integral values of (1).

Observations:

1. $\frac{1}{5} z(a, a)$ is a perfect square
2. $x(a, 1) - 250P_a^5 + 800P_a + G_{31625a} \equiv 0 \pmod 2$
3. $x(1, b) - 49875P_b^5 + 121125P_a - G_{56125b} \equiv 0 \pmod 2$
4. $y(a, 1) + 15P_a^5 + 7785P_a - G_{8172a} \equiv 0 \pmod{11}$
5. $y(1, b) - 98800P_b^5 + 40850P_b - G_{16475b} \equiv 0 \pmod 2$

Each of the following is a nasty number

6. $\frac{6}{5} z(1,0)$, $\frac{6}{25} x(1,0)$, $-\frac{3}{5} y(1,0)$

2.4 *Method: IV*

Instead of (16), write 1 as

$1 = \frac{1}{196}(5 + i3\sqrt{19})(5 + i3\sqrt{19})$ (26)

Using (4), (10) and (26) in (15) and applying the method of factorization, we are found as

$(u + i\sqrt{19}v)(u - i\sqrt{19}v) = \frac{1}{196}[(5 + i3\sqrt{19})(5 - i3\sqrt{19})$
$(2 + i\sqrt{19})(2 - i\sqrt{19})(a + i\sqrt{19}b)^3(a - i\sqrt{19}b)^3]$

It gives us

$(u + i\sqrt{19}v) = \frac{1}{14}(5 + i3\sqrt{19})(2 + i\sqrt{19})(a + i\sqrt{19}b)^3$ (27) $(u - i\sqrt{19}v) = \frac{1}{14}[(5 - i3\sqrt{19})(2 - i\sqrt{19})(a - i\sqrt{19}b)^3]$ (28)

This furnish us

$u = u(a, b) = \frac{1}{14}[-47a^3 - 627a^2b + 2679ab^2 + 361b^3]$ $v = v(a, b) = \frac{1}{14}[112a^3 - 141a^2b - 627ab^2 + 893b^3]$

In sight of (2), the values of x, y are found to be

$x = x(a, b) = \frac{1}{14}[-36a^3 - 768a^2b + 2052ab^2 + 1254b^3]$ (29)

$y = y(a, b) = \frac{1}{14}[-58a^3 - 486a^2b + 3306ab^2 - 1653b^3]$ (30)

Since our intension is to find integer solutions, taking a as 7a and b as 7b in (4), (29) and (30), the related parametric integer values of (1) are found as $x = x(a, b) = -882a^3 - 18816a^2b + 50274ab^2 + 30723b^3$ $y = y(a, b) = -142a^3 - 11907a^2b + 80997ab^2 - 13034b^3$ $z = z(a, b) = 49a^2 + 931b^2$ Hence the above three equations give us two parametric the non-zero different integral values of (1).

Observations:

1. $x(a, a) - y(a, a) + 63798P_a^5 - 31899t_{4,a} = 0$
2. $z(a, a) - t_{394,a} - G_{97a} - P_a - 1 + t_{4,a} = 0$
3. $x(1,b) - 36750P_b^5 + 18375P_b + 18963t_{4,b} + 1911 = 0$
4. $y(a,1) + 4704P_a^5 + 17199P_a + 32634t_{4,a} - 17787 = 0$
5. $z(a, 1) - 49t_{4,a} \equiv 0 \pmod 7$

III. Conclusion

In this work, we have observed various process of determining infinitely a lot of non-zero different integer values to the cubic Diophantine equation $5x^2 + 5y^2 - 9xy = 23z^3$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

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