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On The Non Homogeneous Heptic Equation With Five Unknowns $(x^2 - y^2)(3x^2 + 3y^2 - 5xy) = 5(X^2 - Y^2) z^5$

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Abstract – Five different methods of the non-zero non-negative solutions of non- homogeneous Heptic Diophantine equation $(x^2 - y^2)(3x^2 + 3y^2 - 5xy) = 5(X^2 - Y^2) z^5$ are observed. Introducing the linear transformation $x = u + v$, $y = u - v$, $X = 3u + u$, $Y = 3u - v$, $u \neq v \neq 0$ in $(x^2 - y^2)(3x^2 + 3y^2 - 5xy) = 5(X^2 - Y^2) z^5$, it reduces to $u^2 + 11v^2 = 15z^5$. We are solved the above equation through various choices and the different methods of solutions which are satisfied it. Some interesting relations among the special numbers and the solutions are exposed.

The following notations are used: P_n^m: Pyramid number of rank n with size m- $T_{n,m}$: Polygonal number of rank n with size m- G_a : Gnomonic number of rank a- P_a : Pronic number of rank a- $f^r_{4,3}, f^r_{5,3}$:Fourth and fifth dimensional figurate number of r, whose generating polygon is a Triangle- $f^r_{4,4}, f^r_{5,4}$: Fourth and fifth dimensional figurate number of rank r, whose generating polygon is a Square - $f^r_{4,5}, f^r_{5,5}$: Fourth and fifth dimensional figurate number of rank r, whose generating polygon is a Pentagon – $f^r_{4,6}, f^r_{5,6}$: Fourth and fifth dimensional figurate number of rank r, whose generating polygon is a Hexagon- $f^r_{4,7}, f^r_{5,7}$: Fourth and fifth dimensional figurate number of rank r, whose generating polygon is a Heptagon - $f^r_{4,8}, f^r_{5,8}$: Fourth and fifth dimensional figurate number of rank r, whose generating polygon is a Octagon. 2010 Mathematics Subject Classification: 11D09

Keywords – The Diophantine Equation, Heptic Equation, Integral Solutions, Special Numbers, a Few Interesting Relation.

I. Introduction

The number theory is the queen of Mathematics For an extensive review of variety of problems, one may to[1-23]. In 2014,Jayakumar.P,Sangeetha. K.,[22] have published a paper in finding the integer solutions of the non - homogeneous Heptic Diophantine equation $(x^2 - y^2)(9x^2 + 9y^2 - 16xy) = 21(X^2 - Y^2)z^5$.

In 2015, the same authors Jayakumar**.** P, Sangeetha. K[23] published a paper in finding the integer solutions of the non- homogeneous Heptic Diophantine equation

$(x^2 - y^2)(5x^2 + 5y^2 - 8xy) = 13(X^2 - Y^2)z^5$

Inspired by these, we are observed in this work another interesting five different non-zero integral solutions of the non- homogeneous Heptic Diophantine equation $(x^2 - y^2)(3x^2 + 3y^2 - 5xy) = 5(X^2 - Y^2) z^5$

II. Description of Method

Let us take the Heptic Diophantine equation $(x^2 - y^2)(3x^2 + 3y^2 - 5xy) = 5(X^2 - Y^2)z^5$ (1) We introduce of the linear transformation $x = u + v$, $y = u - v$, $X = 3u + v$, $Y = 3u - v$ (2) Using (2) in (1), gives to $u^2 + 11v^2 = 15z^5$ (3) The equation (3) is solved by the various methods and thus get different solutions to (1)

2.1 Method: I Let us take $z = z(a, b) = a^2 + 11b^2$**,** (4) where a and b are non - zero distinct integers Take 15 as $15 = (2 + i\sqrt{11})(2 - i\sqrt{11})$ (5)

Using (4) and (5) in (3) and applying the factorization process, define $(u + i\sqrt{11}\, v) = (2 + i\sqrt{11})(a + i\sqrt{11}\, b)$ This gives us $u = u(a, b) = 2a^5 - 55a^4b - 220a^3b^2 + 1210a^3b^3 + 1210ab^4 - 1331b^5$

$v = v(a, b) = a^5 + 10a^4b - 110a^3b^2 - 220a^2b^3 + 605ab^4 + 242b^5$ Insight of (2) the relating solutions of (1) are found as $x = x(a, b) = 3a^5 - 45a^4b - 330a^3b^2 + 990a^2b^3 + 1815ab^4 - 1089b^5$

$y = y(a, b) = a^5 - 65a^4b - 110a^3b^2 - 770ab^3 + 605ab^4 - 1573b^5$

$X = X(a, b) = 7a^5 - 155a^4b - 770a^3b^2 + 3410a^2b^3 + 4225ab^4 + 4243b^5$

$Y = Y(a, b) = 5a^5 - 175a^4b - 550a^3b^2 + 3850a^2b^3 + 3025ab^4 + 3751b^5$

$z = z(a, b) = a^2 + 11b^2$

Observations:

1. z (a, 3a) is a perfect number
2. $x(a, 1) - 120 f^a_{5,5} + 210 f^a_{4,6} + 360P_a^5 - 290P_a - G_{749a} \equiv 0 \pmod{17}$
3. $y(a, 1) - 120 f^a_{5,3} + 600 f^a_{4,5} - 105P_a^5 - 150P_a + G_{4a} \equiv 0 \pmod{787}$
4. $X(1, b) - 940 f^a_{5,3} + 1080 f^a_{4,7} + 770 P_a^5 - 752P_a - G_{1601a} \equiv 0 \pmod{13}$
5. $Y(a, 1) - 120 f^a_{5,7} + 984 f^a_{4,7} + 122 P_a^5 - 4196P_a - G_{564a} \equiv 0 \pmod{2}$
6. 2z (1, 1) is a nasty number.

2.2 Method: II

Take (3) as $u^2 + 11v^2 = 15z^5 * 1$ (6)

Let us take 1 as $1 = \frac{1}{36}(5 + i\sqrt{11})(5 - i\sqrt{11})$ (7)

The following techniques is same as in the Method-I, the relating non-zero different integral solutions of (1) give us

$u = u(a, b) = \frac{1}{6}[-a^5 - 385a^4b + 110a^3b^2 + 8470a^2b^3 - 605ab^4 - 9317b^5]$

$v = v(a, b) = \frac{1}{6}[7a^5 - 5a^4b - 770a^3b^2 + 110a^2b^3 + 4235ab^4 - 121b^5]$

In sight of (2), the values of x, y are given by

x = x(a, b) = $\frac{1}{6}[6a^5 - 390a^4b - 660a^3b^2 + 8580a^2b^3 + 3630ab^4 + 9438b^5]$ (8)

y = y (a, b) = $\frac{1}{6}[-8a^5 - 380a^4b + 880a^3b^2 + 8360a^2b^3 4840ab^4 - 9196b^5]$ (9)

X = X(a, b) = $\frac{1}{6}[4a^5 - 1160a^4b - 440a^3b^2 + 25520a^2b^3 + 2420ab^4 - 28072b^5]$ (10)

Y = Y (a, b) = $\frac{1}{6}[-10a^5 - 1150a^4b + 1100a^3b^2 + 25300a^2b^3 + 6050ab^4 - 27830b^5]$ (11)

Since our intension is to find the integer solutions, taking a as 3aand b as 3b in (4), (8),(9), (10) and (11), the relating two parametric integer solutions of (1) are found as

x = x (a, b) = $243a^5 - 15795a^4b - 2673b^2 + 347490a^2b^3 + 147015ab^4 + 382239b^5$

y = y (a, b)=$324a^5 - 15390a^4b + 71280a^3b^2 + 677160a^2b^3 - 382040ab^4 - 744876b^5$

X = X (a, b)=$[162a^5 - 46980a^4b - 17820a^3b^2 + 993060a^2b^3 + 98010ab^4 - 1136916b^5]$

Y =Y(a, b) = $-405a^5 - 46575a^4b + 44550a^3b^2 + 1024650a^2b^3 - 245025ab^4 - 1127115b^5$

z = z (a, b) = $9a^2 + 99b^2$

Observations:

1. z (a,0) is a perfect number
2. x (a, 1) - 9720 $f_{5,5}^{a}$ +83592 $f_{4,7}^{a}$ +2512P_a^5- 381187 P_a +G_{12550a}≡ 0 (mod 2)
3. y (a, 1) + 9720 $f_{5,6}^{a}$ + 64200 $f_{4,7}^{a}$ - 225560P_a^5- 585940P_a + $G_{486072a}$≡ 0 (mod 5)
4. X (1, a) +34107480 $f_{5,6}^{a}$ -34577928 $f_{4,7}^{a}$ +29931896 P_a^5-14810914P_a–$G_{66455513a}$≡ 0 (mod 7)
5. Y (a, 1) + 9720 $f_{5,7}^{a}$ + 211896 $f_{4,7}^{a}$ -33082 P_a^5 – 1070155P_a +$G_{666919a}$≡ 0 (mod 2)
6. $\frac{2}{9}$ z(1,1) is a nasty number.

2.3: Method: III

In place of (5) take 15 as

$15 = \frac{1}{4}(7 + i\sqrt{11})(7 - i\sqrt{11})$ (8)

The following techniques is same as in the Method-I, and doing some calculations, the relating the non-zero different integral solutions of (1) give us

u = u (a, b) = $\frac{1}{2}[7a^5 - 55a^4b - 770a^3b^2 + 121a^2b^3 + 4235ab^4 - 1331b^5]$

v = v (a, b) = $\frac{1}{2}[a^5 + 35a^4b - 110a^3b^2 - 770a^2b^3 + 605ab^4 + 847b^5]$

In sight of (2), the integer values of x and y are

x = x (a, b) = $4a^5 - 10a^4b - 440a^3b^2 + 20a^2b^3 + 2420ab^4 - 242b^5$

y = y(a, b) = $3a^5 - 45a^4b - 330a^3b^2 + 990a^2b^3 + 1815ab^4 - 1089b^5$

X = X(a, b) = $11a^5 - 65a^4b - 1210a^3b^2 + 2200a^2b^3 + 6655ab^4 - 2662b^5$

Y = Y(a, b) = $10a^5 - 100a^4b - 1100a^3b^2 + 1430a^2b^3 60500ab^4 - 2420b^5$

z = z (a, b) = $a^2 + 11b^2$

Observations:

1. $\frac{2}{9}$ z (1, 1) is a nasty number
2. z (a, a) -12$t_{4,a}$ = 0
3. 3x (a, 1) –4 y(a, 1) -150 $t_{4,a}^2$+330 $t_{4,a}$≡ 0 (mod2)
4. 3x (a, 1) – 4 y(a, 1) ≡ 0 (mod2)
5. 10X (a, 1) -11Y (a, 1)- 450 $t_{4,a}^2$ -13530 $t_{4,a}$ = 0
6. 10X(a, 0) -11Y (a, 1) = 0
7. z (a, 0) –$t_{4,a}$ = 0

2.4: Method: 4

In place of (7) take 1 as

$1 = \frac{1}{100}(1 + i3\sqrt{11})(7 - i3\sqrt{11})$ (8)

The following technique is same as in the Method-I, and doing some calculations, the relating the non-zero different integral solutions of (1) give us

u = u(a, b) = $\frac{1}{10}[a^5 - 165a^4b - 110a^3b^2 3630a^2b^3 + 605ab^4 - 3993b^5]$

v = v (a, b)= $[3a^5 + 5a^4b - 330a^3b^2 - 110a^2b^3 + 1815ab^4 + 121b^5]$

In true of (4), the values of x and y are

x = x (a, b) = $\frac{1}{10}[6a^5 - 390a^4b - 660a^3b^2 + 8580a^2b^3 + 3630ab^4 + 9438b^5]$ (8)

y = y (a, b) = $\frac{1}{10}[-8a^5 - 380a^4b + 880a^3b^2 + 8360a^2b^3 - 4840ab^4 - 9196b^5]$ (9)

X = X (a, b) = $\frac{1}{10}[4a^5 - 1160a^4b - 440a^3b^2 + 25520a^2b^3 + 2420ab^4 - 28072b^5]$ (10)

Y = Y (a, b) = $\frac{1}{10}[-10a^5 - 1150a^4b + 1100a^3b^2 + 25300a^2b^3 - 6050ab^4 - 27830b^5]$ (11)

Since our intension is to find the integer solutions, taking a as 3a and b as 3b in (4), (8), (9), (10) and (11), the relating two parametric integer solutions of (1) are found as

x = x (a, b) = $1250a^5 - 50000a^4b - 137500a^3b^2 + 1100000a^2b^3 - 756250ab^4 - 1210000b^5$,

y= y (a, b) = $-625a^5 - 53125a^4b - 68750a^3b^2 + 1137500a^2b^3 - 40625ab^4 - 1285625b^5$

X = X (a, b) = $1875a^5 - 153125a^4b - 206250a^3b^2 + 3362500a^2b^3 + 408750b^5$

Y= $Y(A, B) = -156250a^4b + 3437500a^2b - 580625b^5$

z = z (a, b) = $25a^2 + 275b^2$

Observations:

1. $\frac{1}{10}$ z (1, 1) is a nasty number
2. z(a, a) -300 $t_{4,a}$ = 0
3. x (a, 1) +2 y(a, 1) + 150250 $t_{4,a}^2$- 3375000 $t_{4,a}$ + $G_{3375000a}$≡ 1 (mod2)
4. x (a, 0) +2 y(a, 0) = 0
5. X(a, 1) +3Y (a, 1)+31250 $t_{4,a}^2$-6775000 $t_{4,a}$ +G_{60932a}≡ 0 (mod 11)
6. X(a, 0) +3Y (a, 0) ≡ 0 (mod 5)
7. z (a, 0) – 25$t_{4,a}$ = 0
8. Y (a, 1) + 156250 $t_{4,a}^2$ -3437500 $t_{4,a}$ ≡ 0 (mod 5)

2.3: Method: 5

Instead of (5) write 15 as

\ $15 = \frac{1}{36}(1 + i7\sqrt{11})(1 - i7\sqrt{11})$ (12)

Following the procedure similar to Pattern-III, and performing a few calculations, the corresponding non-zero distinct integral solutions of (1) are found to be

x = x (a, b) = $\frac{1}{6}[6a^5 - 390a^4 b - 660a^3b^2 + 8580a^2b^3 + 3630ab^4 + 9438b^5]$ (13)

y = y (a, b) = $\frac{1}{6}[-8a^5 - 380a^4b + 880a^3b^2 + 8360a^2b^3 - 4840ab^4 - 9196b^5]$ (14)

X = X (a, b) = $\frac{1}{6}[4a^5 - 1160a^4b - 440a^3b^2 + 25520a^2b^3 + 2420ab^4 - 28072b^5]$ (15)

Y = Y (a, b) = $\frac{1}{6}[-10a^5 - 1150a^4b + 1100a^3b^2 + 25300a^2b^3 - 6050ab^4 - 27830b^5]$ (16)

As our intension is to find the integer solutions, taking a as 3a and b as 3b in (4), (13), (14), (15) and (16), the relating two parametric integer solutions of (1) are found as

x = x (a, b) = $10368a^5 - 492480a^4b11404800a^3b^2 + 10842336a^2b^3 + 6272640ab^4 - 11918016b^5$

y = y (a, b) = $-7776a^5 - 505440a^4b + 8553600a^3b^2 + 11127456a^2b^3 + 4704480ab^4 - 12231648b^5$

X = X (a, b) = $11664a^5 - 991440a^4b - 1283040a^3b^2 + 21827232a^2b^3 + 7056720ab^4 - 23992848b^5$

Y = Y (a, b) = $-6480a^5 - 1004400a^4b - 712800a^3b^2 + 22112352a^2b^3 - 3920400ab^4 - 24306480b^5$

z = z (a, b) = $36a^2 + 396b^2$

Observations:

1. z (1, 0) is a perfect number
2. z (a, a) - $432t_{4,a} = 0$
3. 3x (a, 1) – 4 y (a, 1) -150 $t_{4,a}^2$+330 $t_{4,a} \equiv 0$ (mod2)
4. 3x (a, 1) – 4 y (a, 1) ≡ 0 (mod2)
5. 10X (a, 1) -11Y (a, 1) - 450 $t_{4,a}^2$ -13530 $t_{4,a} = 0$
6. 10X (a, 0) -11Y (a, 1) = 0
7. z (a, 0) – $t_{4,a} = 0$

III. Conclusion

In this work, we have observed various process of determining infinitely a lot of non-zero different integer values to the non-homogeneous Heptic Diophantine equation$(\mathbf{x^2 - y^2})(3x^2 + 3y^2 - 5xy) = 5(X^2 - Y^2)z^5$. One may try to find non-negative integer solutions of the above equations together with their similar observations.

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