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Solution to Partial Fractional Differential Equations with Moving Boundaries in Two Dimensions

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***Abstract* – In this article, we implement Laplace transform method for the solution of certain partial fractional differential equation with moving boundaries in two dimensions. The results reveal that the transform method is very effective and convenient. Illustrative examples are also provided.**
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I. Introduction and Preliminaries

In the present study, the fractional derivatives are understood in the Caputo sense. The reason for adopting the Caputo definition is as follows: There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann-Liouville, Grünwald–Letnikov, Caputo and generalized functions approach [5,6].Riemann-Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.
For an arbitrary real number

$$\alpha > 0 \ \ (n-1 \le \alpha < n\ ,\ n \in N\ )$$

Caputo fractional derivative is given as

$$ {}_a^C D_t^\alpha f(t) = {}_a I_t^{n-\alpha} f^{(n)}(t) $$
$$ = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx . $$

The direct Laplace transform of a function $f(t)$ defined for $0 \le t < \infty$ is the ordinary calculus integration problem

$$ L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt := F(s). $$

If $L\{f(t)\} = F(s)$, then $L^{-1}\{F(s)\}$ is given by

$$ f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ts} F(s)\, ds \quad (t > 0) $$

where $F(s)$ is analytic in the region $\mathrm{Re}(s) > c$ and $f(t) = 0$ for $t < 0$.

This result is called complex inversion formula. It is also known as Bromwich's integral formula. The one-dimensional convolution theorem of $f(x)$ and $g(x)$ is given by

$$ f(x) * g(x) = \int_0^x f(x-w)\, g(w)\, dw . $$

The Laplace transform of Caputo’s fractional derivative is given by [4]

$$ L\{ {}_0^C D_t^\alpha u(x,t) \} $$
$$ = s^\alpha U(x,s) - \sum_{r=0}^{n-1} s^{\alpha-r-1} u^{(r)}(x,0) $$
$$ (n-1 < \alpha \le n). $$

The simplest Wright function is defined by the series [4]

$$ W(\alpha,\beta;z) := \sum_{k=0}^\infty \frac{z^k}{k!\,\Gamma(\alpha k+\beta)} $$
$$ (\alpha,\beta,z \in C). $$

Example 1.1: The Laplace transform inversion of function $K_n(as)$ (modified Bessel function of order n) is obtained by integral representation of this function as following

$$ L^{-1}\{K_n(as)\} = L^{-1}\left\{ \int_0^\infty e^{-as\cosh u} \cosh(nu)\, du \right\} $$
$$ = \int_0^\infty \cosh(nu) L^{-1}\{ e^{-(a\cosh u)s} \} du $$
$$ = \int_0^\infty \cosh(nu)\, \delta(t - a\cosh u)\, du , $$

where δ is Dirac delta function. We know that

$$ \int_a^b f(x)\,\delta(x-w)\, dx = f(w), $$

then we can use this formula to get

$$ \int_0^\infty \cosh(nu)\,\delta(t-a\cosh u)\, du $$
$$ = \cosh(n \cosh^{-1}\frac{t}{a}). $$

So that

$$ L^{-1}\{K_n(as)\} = \int_0^\infty \cosh(nu)\,\delta(t-a\cosh u)\, du $$
$$ = \cosh(n\cosh^{-1}\frac{t}{a}). $$

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Theorem 1.1: **(Schouten-Van der Pol Theorem)** Consider $L\{f(t)\}=F(s)$ which $F(s)$ is analytic in the half-plane $\mathrm{Re}(s)>s_0$. We can use this knowledge to find $g(t)$ whose Laplace transform $G(s)$ equals $F(\varphi(s))$, where $\varphi(s)$ is also analytic for $\mathrm{Re}(s)>s_0$. This means that if

$$G(s)=F(\varphi(s))=\int_0^\infty f(\tau)\exp(-\varphi(s)\tau)d\tau$$

and

$$g(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(\varphi(s))\exp(ts)ds,$$

therefore

$$g(t)=\int_0^\infty f(\tau)L^{-1}\{e^{-\tau\varphi(s)}\}d\tau.$$

Proof. See [3].

Example 1.2: We can use the above theorem for computation of Laplace transform inversion

$$L^{-1}\left\{\frac{1}{(s^\alpha+a)(s^\beta+b)}\right\}\quad(0<\alpha,\beta<1).$$

We know that

$$L^{-1}\{\frac{1}{s+a}\}=e^{-at},$$

Applying the Schouten-Vanderpol theorem gives

$$L^{-1}\{\frac{1}{s^\alpha+a}\}=\int_0^\infty e^{-a\tau}L^{-1}\{e^{-\tau s^\alpha}\}d\tau,$$

that by expansion of $e^{-\tau s^\alpha}$, we get

$$L^{-1}\{\frac{1}{s^\alpha+a}\}=\int_0^\infty e^{-a\tau}\sum_{n=0}^\infty\frac{(-\tau)^n}{n!}L^{-1}\{\frac{1}{s^{-n\alpha}}\}d\tau$$

$$=\int_0^\infty e^{-a\tau}\sum_{n=0}^\infty\frac{(-\tau)^n t^{-\alpha n-1}}{n!\Gamma(-\alpha n)}d\tau,$$

since series in integral is uniformly convergence, we can write

$$L^{-1}\{\frac{1}{s^\alpha+a}\}=\sum_{n=0}^\infty\frac{(-1)^n t^{-\alpha n-1}}{n!\Gamma(-\alpha n)}\int_0^\infty e^{-a\tau}\tau^n d\tau$$

$$=\sum_{n=0}^\infty\frac{(-1)^n t^{-\alpha n-1}}{\Gamma(-\alpha n)a^{n+1}}$$

$$=\frac{1}{at}\sum_{n=0}^\infty\frac{(-\frac{1}{at^\alpha})^n}{\Gamma(-\alpha n)}=\frac{1}{at}E_{-\alpha,0}(-\frac{1}{at^\alpha}).$$

Similarly,

$$L^{-1}\{\frac{1}{s^\beta+b}\}=\frac{1}{bt}E_{-\beta,0}(-\frac{1}{bt^\beta}).$$

Then, using the convolution theorem gives

$$L^{-1}\left\{\frac{1}{(s^\alpha+a)(s^\beta+b)}\right\}$$

$$=\frac{1}{ab}\int_0^t\left(\frac{1}{\eta}E_{-\alpha,0}(-\frac{1}{a\eta^\alpha})\right)$$

$$\times\left(\frac{1}{t-\eta}E_{-\beta,0}(-\frac{1}{b(t-\eta)^\beta})\right)d\eta.$$

Theorem 1.2: Let $f(t)$ denote a real-valued function, where its Laplace transform $F(s)$ exists. Let $F(s)$ satisfy the following hypothesis:

1) $F(s)$ is a multi-valued function which has no singularities in the cut s −plane. The branch cut lies along the negative real axis $(-\infty,0]$.

2) $F^*(s)=F(s^*)$, where the star denotes the complex conjugate.

3) $F^\pm(\eta)=\lim_{\varphi\to\pi^-}F(\eta e^{\pm\varphi i})$ and

$F^+(\eta)=(F_-(\eta))^*$.

4) $F(s)=o(1)$ as $|s|\to\infty$ and $F(s)=o(\frac{1}{|s|})$ as $|s|\to 0$, uniformly in any sector $|\arg(s)|<\pi-\eta$, $0<\eta<\pi$.

5) There exists $\varepsilon>0$, such that for every $\pi-\varepsilon<\varphi\le\pi$, $\frac{F(re^{\pm\varphi i})}{1+r}\in L_1(R^+)$ and $|F(re^{\pm\varphi i})|<a(r)$, where $a(r)$ does not depend on φ and $a(r)e^{-\delta r}\in L_1(R^+)$ for any $\delta>0$.

Then

$$f(t)=\frac{1}{\pi}\int_0^\infty \mathrm{Im}(F^-(\eta))e^{-t\eta}d\eta.$$

Proof. See [3].

Some interesting applications of the above theorem are represented in following.

Lemma 1.1: The fractional partial differential equation

$${}_0^C D_t^\alpha u(x,t)=c\frac{\partial u(x,t)}{\partial x}+bu(x,t),$$

where $0<\alpha<1$, $t>0$, $-\infty<x<\infty$, $c\neq 0$ and b are constant and with boundary condition $u(x,0)=f(x)$, has the following formal solution

$$u(x,t)$$

$$=\frac{1}{t}\int_0^\infty e^{b\tau}W(-\alpha,0;-\tau t^{-\alpha})f(x+c\tau)d\tau.$$

Proof: By taking the Laplace transform of both sides of the fractional equation and assuming that $L\{u(x,t)\}=U(x,s)$, we have

$$s^\alpha U(x,s)-f(x)=cU'(x,s)+bU(x,s).$$

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Applying the Fourier transform with respect to x gives

$$s^{\alpha}\overline{U}(w,s)-F(w)$$
$$=-c\,i\,\overline{U}(w,s)+b\overline{U}(w,s),$$

or

$$\overline{U}(w,s)=\frac{F(w)}{s^{\alpha}+i\,w-b}.$$

Using the Schouten-Van der Pol theorem, we have

$$L^{-1}\{\frac{1}{s^{\alpha}+c\,i\,w-b}\}$$
$$=\frac{1}{t}\int_0^{\infty}e^{-c\tau\omega i}e^{b\tau}W(-\alpha,0;-\tau t^{-\alpha})d\tau,$$

Therefore, by applying the Laplace transform inversion

$$\bar{u}(\omega,t)=L^{-1}\{\frac{F(\omega)}{s^{\alpha}+c\,i\,w-b}\}$$
$$=\frac{F(\omega)}{t}\int_0^{\infty}e^{-c\tau\omega i}e^{b\tau}W(-\alpha,0;-\tau t^{-\alpha})d\tau.$$

Now, using the Fourier transform inversion and convolution theorem in this transform gives

$$u(x,t)$$
$$=F^{-1}\left\{\frac{F(\omega)}{t}\int_0^{\infty}e^{-c\tau\omega i}e^{b\tau}W(-\alpha,0;-\tau t^{-\alpha})d\tau\right\}$$
$$=\frac{1}{t}\int_0^{\infty}e^{b\tau}W(-\alpha,0;-\tau t^{-\alpha})f(x+c\tau)d\tau.$$

II. Evaluation of the Integrals

In applied mathematics, the **Kelvin functions** $Ber_{\nu}(x)$ and $Bei_{\nu}(x)$ are the real and imaginary parts, respectively, of $J_{\nu}(xe^{3\pi i/4})$, where x is real, and $J_{\nu}(z)$ is the ν-th order Bessel function of the first kind. Similarly, the functions $Ker_{\nu}(x)$ and $Kei_{\nu}(x)$ are the real and imaginary parts, respectively, of $K_{\nu}(xe^{\pi i/4})$, where $K_{\nu}(z)$ is the ν-th order modified Bessel function of the second kind. These functions are named after William Thomson, 1st Baron Kelvin. The Kelvin functions were investigated because they are involved in solutions of various engineering problems occurring in the theory of electrical currents, elasticity and in fluid mechanics.

One of the main applications of Laplace transform is evaluating the integrals as discussed in the following.

Lemma 2.1. The following integral relationship holds true

$$\frac{2}{\pi}\int_0^{\infty}\frac{ber(\sqrt{2\lambda})d\lambda}{\lambda^2-\xi^2}=\frac{1}{\xi}bei(2\sqrt{\xi}).$$

Proof: Let us define the following function

$$I(t)=\frac{2}{\pi}\int_0^{\infty}\frac{ber(\sqrt{2t\lambda})d\lambda}{\lambda^2-\xi^2}.$$

Laplace transform of $I(t)$ yields

$$L\{I(t)\}=\int_0^{\infty}e^{-st}\left(\frac{2}{\pi}\int_0^{\infty}\frac{ber(\sqrt{2t\lambda})d\lambda}{\lambda^2-\xi^2}\right)dt,$$

changing the order of integration, which is permissible, leads to

$$L\{I(t)\}$$
$$=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\lambda^2-\xi^2}(\int_0^{\infty}e^{-st}ber(\sqrt{2\lambda t})dt)d\lambda,$$

or,

$$L\{I(t)\}=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\lambda^2-\xi^2}(\frac{1}{s}\cos\frac{\lambda}{2s})d\lambda.$$

After simplifying, we obtain

$$L\{I(t)\}=\frac{1}{\pi s}\int_{-\infty}^{\infty}\frac{\cos\frac{\lambda}{2s}}{\lambda^2-\xi^2}d\lambda.$$

At this point, by using table of integrals or residue theorem, we get the following

$$L\{I(t)\}=\frac{1}{\pi s}\{\frac{\pi}{2\xi}\sin\frac{\xi}{s}\}=\frac{1}{2\xi s}\sin\frac{\xi}{s}.$$

Taking inverse Laplace transform of the above relationship, we get finally

$$I(t)=L^{-1}\{\frac{1}{2\xi s}\sin\frac{\xi}{s}\}=\frac{1}{2\xi}bei(2\sqrt{\xi t}).$$

Letting $t=1$, we get

$$\frac{2}{\pi}\int_0^{\infty}\frac{ber(\sqrt{2\lambda})d\lambda}{\lambda^2-\xi^2}=\frac{1}{\xi}bei(2\sqrt{\xi}).$$

III. Partial Fractional Differential Equation Moving Boundary in Two Dimensions

Let us solve fractional partial differential equation

$${}_0^C D_t^{\alpha}u(x,y,t)=a^2(u_{xx}+u_{yy}),$$

where $0<\alpha\le 1$ with initial and boundary conditions

$$\lim_{x\to\infty}u=\lim_{y\to\infty}u=0,$$

$$u\Big|_{x=\beta t,\,y=\gamma t}=\frac{1}{\sqrt{\pi t}}e^{-\frac{1}{4t}},$$

$$u(x,y,0)=0.$$

With change of variables $\eta=x-\beta t$ and $\xi=y-\gamma t$, we obtain

$$ {}_0^C D_t^{\alpha} w(\eta,\xi,t) - (\beta\frac{\partial}{\partial\eta}+\gamma\frac{\partial}{\partial\xi})\,{}_0I_t^{1-\alpha} w(\eta,\xi,t) = a^2(w_{\eta\eta}+w_{\xi\xi}), $$

$$ \lim_{\eta\to\infty} w = \lim_{\xi\to\infty} w = 0\ ,\ w(\eta,\xi,0)=0, $$

$$ w(0,0,t)=\frac{1}{\sqrt{\pi t}}e^{-\frac{1}{4t}}. $$

By applying the Laplace transform and condition $w(\eta,\xi,0)=0$, we will get

$$ s^{\alpha}W(\eta,\xi,s)-(\beta\frac{\partial}{\partial\eta}+\gamma\frac{\partial}{\partial\xi})\frac{1}{s^{1-\alpha}}W(\eta,\xi,s) = a^2(W_{\eta\eta}+W_{\xi\xi}). $$

By using the separation of variable method as $U=A(\eta,s)B(\xi,s)$, we obtain

$$ s^{\alpha}-\frac{\beta}{s^{1-\alpha}}\frac{A'_{\eta}}{A}-a^2\frac{A''_{\eta\eta}}{A}=a^2\frac{B''_{\xi\xi}}{B}+\frac{\gamma}{s^{1-\alpha}}\frac{B'_{\xi}}{B}=\lambda, $$

where $\lambda>0$. Then

$$ A=c_1 e^{-\frac{\beta\eta}{2a^2 s^{1-\alpha}}} e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^{\alpha}-\lambda)}} + c_2 e^{-\frac{\beta\eta}{2a^2 s^{1-\alpha}}} e^{\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^{\alpha}-\lambda)}}, $$

$$ B=c'_1 e^{-\frac{\gamma\xi}{2a^2 s^{1-\alpha}}} e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} + c'_2 e^{-\frac{\gamma\xi}{2a^2 s^{1-\alpha}}} e^{\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}}. $$

Using the conditions $\lim_{\eta\to\infty} w = \lim_{\xi\to\infty} w = 0$ gives $c_2=c'_2=0$, therefore we obtain

$$ W(\eta,\xi,s)=\int_0^{\infty} c_0(s,\lambda) e^{-\frac{\beta\eta+\gamma\xi}{2a^2 s^{1-\alpha}}} e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} \times e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^{\alpha}-\lambda)}}\,d\lambda. $$

Laplace transform of the boundary condition

$$ w(0,0,t)=\frac{1}{\sqrt{\pi t}}e^{-\frac{1}{4t}} $$

leads to

$$ W(0,0,s)=\int_0^{\infty} c_0(s,\lambda)\,d\lambda $$

$$ =\frac{e^{-\sqrt{s}}}{\sqrt{s}}=\int_0^{\infty} e^{-s\lambda}\frac{e^{-\frac{1}{4\lambda}}}{\sqrt{\pi\lambda}}\,d\lambda. $$

We can conclude that $c_0(s,\lambda)=\frac{e^{-s\lambda-\frac{1}{4\lambda}}}{\sqrt{\pi\lambda}}$. By replacing $c_0(s,\lambda)$, we have

$$ W(\eta,\xi,s)=\int_0^{\infty} e^{-s\lambda}\frac{e^{-\frac{1}{4\lambda}}}{\sqrt{\pi\lambda}} e^{-\frac{\beta\eta+\gamma\xi}{2a^2 s^{1-\alpha}}} \times e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^{\alpha}-\lambda)}}\,d\lambda. $$

Now we obtain $u(x,y,t)$ by inverting Laplace transform

$$ u(\eta,\xi,t)=L^{-1}\{W(\eta,\xi,s)\} $$

$$ \int_0^{\infty}\frac{e^{-\frac{1}{4\lambda}}}{\sqrt{\pi\lambda}} L^{-1}\left\{ e^{-s\lambda} e^{-\frac{\beta\eta+\gamma\xi}{2a^2 s^{1-\alpha}}} e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} \times e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^{\alpha}-\lambda)}} \right\} d\lambda. $$

At this point, we can use the theorem 2.1 for obtaining inversion of inner term. First, we set

$$ F(s)=e^{-s\lambda} e^{-\frac{\beta\eta+\gamma\xi}{2a^2 s^{1-\alpha}}} e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} \times e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^{\alpha}-\lambda)}}, $$

then

$$ F_{-}(\rho)=\lim_{\varphi\to\pi} F(\rho e^{-i\varphi}) = $$

$$ e^{-\lambda\rho e^{-i\pi}} e^{-\frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}e^{-(1-\alpha)\pi i}}} \times e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{\rho^{2-2\alpha}e^{-(2-2\alpha)\pi i}}+4a^2\lambda}} \times e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{\rho^{2-2\alpha}e^{-(2-2\alpha)\pi i}}+4a^2(\rho^{\alpha}e^{-\alpha\pi i}-\lambda)}} $$

$$ = e^{\lambda\rho+\frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}}\cos\alpha\pi} e^{-i\frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}}\sin\pi\pi} \times e^{-\frac{\xi}{2a^2}\sqrt{(\frac{\gamma^2}{\rho^{2-2\alpha}}\cos 2\alpha\pi+4a^2\lambda)-i\frac{\gamma^2}{\rho^{2-2\alpha}}\sin 2\alpha\pi}} $$

$$ \times\exp\left\{-\frac{\eta}{2a^2}\left((\frac{\beta^2}{\rho^{2-2\alpha}}\cos 2\alpha\pi+4a^2\rho^{\alpha}\cos\alpha\pi - a^2\lambda)-i(\frac{\beta^2}{\rho^{2-2\alpha}}\sin 2\alpha\pi+4a^2\rho^{\alpha}\sin\alpha\pi)\right)^{\frac{1}{2}}\right\} $$

$$= e^{\lambda\rho + \frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}}\cos\alpha\pi} e^{-i\frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}}\sin\pi\pi}$$

$$\times e^{-\frac{\xi}{2a^2}\sqrt{r_1}e^{i\frac{\theta_1}{2}}} e^{-\frac{\eta}{2a^2}\sqrt{r_2}e^{i\frac{\theta_2}{2}}},$$

where

$$r_1 = \left((\frac{\gamma^2}{\rho^{2-2\alpha}}\cos 2\alpha\pi + 4a^2\lambda)^2 + \frac{\gamma^4}{\rho^{4-4\alpha}}\sin^2 2\alpha\pi \right)^{\frac{1}{2}},$$

$$\theta_1 = -\tan^{-1}\left(\frac{\frac{\gamma^2}{\rho^{2-2\alpha}}\sin 2\alpha\pi}{\frac{\gamma^2}{\rho^{2-2\alpha}}\cos 2\alpha\pi + 4a^2\lambda} \right),$$

$$r_2 = \left((\frac{\beta^2}{\rho^{2-2\alpha}}\cos 2\alpha\pi + 4a^2\rho^\alpha\cos\alpha\pi - 4a^2\lambda)^2 + (\frac{\beta^2}{\rho^{2-2\alpha}}\sin 2\alpha\pi + 4a^2\rho^\alpha\sin\alpha\pi)^2 \right)^{\frac{1}{2}}$$

$$\theta_2 = -\tan^{-1}\left(\frac{\frac{\beta^2}{\rho^{2-2\alpha}}\sin 2\alpha\pi + 4a^2\rho^\alpha\sin\alpha\pi}{\frac{\beta^2}{\rho^{2-2\alpha}}\cos 2\alpha\pi + 4a^2\rho^\alpha\cos\alpha\pi - 4a^2\lambda} \right).$$

Imaginary part of $F_{\_}(\rho)$ is as following

$$\mathrm{Im}(F_{\_}(\rho)) = -e^{\lambda\rho + \frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}}\cos\alpha\pi - \frac{\xi}{2a^2}\sqrt{r_1}\cos\frac{\theta_1}{2} - \frac{\eta}{2a^2}\sqrt{r_2}\cos\frac{\theta_2}{2}}$$

$$\times\sin\left(\frac{\beta\eta+\gamma\xi}{2a^2\rho^{1-\alpha}}\sin\alpha\pi + \frac{\xi}{2a^2}\sqrt{r_1}\sin\frac{\theta_1}{2} + \frac{\eta}{2a^2}\sqrt{r_2}\sin\frac{\theta_2}{2} \right).$$

According to the theorem 2.1, the final solution is given by

$$L^{-1}\left\{ e^{-s\lambda} e^{-\frac{\beta\eta+\gamma\xi}{2a^2 s^{1-\alpha}}} e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} \times e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^\alpha-\lambda)}} \right\}$$

$$= \frac{1}{\pi}\int_0^\infty \mathrm{Im}(F_{\_}(\rho))e^{-t\rho}\,d\rho.$$

Then

$$u(\eta,\xi,t) = \int_0^\infty \frac{e^{-\frac{1}{4\lambda}}}{\sqrt{\pi\lambda}} L^{-1}\left\{ e^{-s\lambda} e^{-\frac{\beta\eta+\gamma\xi}{2a^2 s^{1-\alpha}}} \times e^{-\frac{\xi}{2a^2}\sqrt{\frac{\gamma^2}{s^{2-2\alpha}}+4a^2\lambda}} e^{-\frac{\eta}{2a^2}\sqrt{\frac{\beta^2}{s^{2-2\alpha}}+4a^2(s^\alpha-\lambda)}} \right\} d\lambda$$

$$= \int_0^\infty \frac{e^{-\frac{1}{4\lambda}}}{\sqrt{\pi\lambda}} \left(\frac{1}{\pi}\int_0^\infty \mathrm{Im}(F_{\_}(\rho))e^{-t\rho}\,d\rho \right) d\lambda.$$

IV. Conclusion

Laplace transform techniques are applied to the solution of moving boundary problems which are governed by diffusion processes such as heat and mass transfer. Freezing and diffusion with a rapid chemical reaction are examples of such problems.

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