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Scattering and Homogenization with Small Dipole

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Abstract – **We consider homogenization questions related to Maxwell partial differential equations. More precisely, we consider an assembly of particles, viewed as dipoles, separated in a periodic way by a typical length 2ε. The size of each dipole is denoted by r_ε. Depending on the relative size of these two scales, we describe the limit partial differential equations, showing a typical discontinuity in the effective frequency.**

Keywords – **Somerfield's Condition, Helmholtz Equation, Limiting Absorption Principle, Maxwell's Equations, Homogenization, Two-Scale Convergence, Oscillating Test Functions.**

I. Introduction

In this paper, we consider a diffraction problem with a given fixed frequency ω. It can be described as follows. In $\mathbb{R}^3$, we consider a finite number N of balls in a domain Ω of $\mathbb{R}^3$ and an electromagnetic wave hitting the balls. These balls can be interpreted as small particles or more generally as small heterogeneities. Numerous works were made to study optic metallic particles properties and the first work was made by Garnett Maxwell [8]. He considered the passage of light through a dielectric medium that contains a lot of small metallic spheres in a volume comparable to a wave length. With the help of the Lorentz-Lorenz formula, Garnett Maxwell showed that such an assembly is equivalent to a mean medium with a certain complex refractive index.

In a paper published in 1908, G.Mie [8, 10] got, using electromagnetic theory, an explicit solution for the diffraction of a plane wave by a homogeneous sphere. He also treated the case of several metallic spheres provided that they are all of the same diameter and of same composition [8, 10, 11]. All in all, all these works deal with the assembly diffraction problem by looking for explicit solutions for Maxwell's equations, and thus are limited to perfect conductor boundary conditions.

In this paper, we shall use the mathematical variational method to study this problem and without looking for an explicit solution, since we shall also use other boundary conditions, of Calderon type, which are closer to our physical interpretation.

In fact, let us mention that the above mentioned works also deal with high frequency length ω, so depending also on the length of each particle. Here we assume that this frequency is fixed, and planned to deal with this problem later, since this is a much more complex situation.

Let Ω be a bounded regular open set of $\mathbb{R}^3$. In Ω, we consider a given assembly of N particles and outside Ω, an incident (on $\partial\Omega$) electromagnetic field with a fixed frequency $\omega > 1$ (see figure 1 at the end of paper).

Recall that, while solving Maxwell's equations with a frequency ω, we are led to an electric field E and a magnetic field B, the pair being a solution of the stationary Maxwell's system see [21].

This solution can be written as the sum of two kinds of waves, the reflected wave, living outside the domain Ω, and the transmitted wave, living in the interior of the domain see [13,14,15,17].

In this paper, we are interested in the asymptotic behavior of the reflected and transmitted waves, when the number of particles N goes to infinity. In the domain Ω, we assume that each particle behaves as an electric dipole as regards to the incident wave. Each electric dipole depends on the dipolar moment of the particle, itself depending on the electromagnetic field *E, B*. Let us introduce some notations to simplify our problem.

We denote by μ_0 and ε_0 the magnetic permeability and electric permittivity (assumed to be constant). $P \in C^3$ stands for the dipolar momentum and δ_0 for Dirac distribution at zero.

II. Microscopic Problem for One Particle

We shall first consider that our assembly of particles is reduced to a single particle. In this case, it is possible to deduce scattering laws in the vicinity of this particle. It yields a modification by equivalent boundary conditions. Then we will be able to consider our full assembly of particles. Most of the computations below can be found in classical books on electromagnetism [15, 18,19], but we have used heavily a presentation due to Cessenat [12]. We will need the following definition.

Definition (1). We call outgoing (resp. incoming) Sommerfeld's condition at infinity the following condition on u

$$\frac{\partial u}{\partial r} - iku = o(\frac{1}{r}).$$

$$\text{(resp.)}\frac{\partial u}{\partial r} + iku = o(\frac{1}{r}). \qquad (1.1)$$

for a given vector valued function u, with o uniform with respect to $\vec{\alpha} = \frac{\vec{x}}{r}$.

An outgoing (resp. incoming) wave is a (vector) solution u of the Helmholtz equation in the whole space

$$\Delta u + k^2 u = -f \text{ in } \mathbb{R}^3,$$

with $k > 0$ and f a distribution with compact support, and u satisfying the outgoing (resp. incoming) Sommerfeld's condition, for more details see [3,4,5,12].

Now, we assume herein that we have only one dipole, centered at the origin with a dipolar momentum P (see figure 2 at the end of paper). Thus the total electromagnetic field *E, B* is solution of Maxwell's problem

$$\begin{cases} a) \quad \mathrm{rot}H + i\omega\epsilon_0 E = J, \qquad J = i\omega P\delta_0 \qquad \text{in} \qquad \mathbb{R}^3, \\ b) \quad -\mathrm{rot}E + i\omega\mu_0 H = 0 \qquad \text{in} \qquad \mathbb{R}^3. \end{cases} \tag{1.3}$$

Classically, we also impose Silver-Muller radiation condition at infinity (outgoing wave condition)

$$\omega\epsilon_0\alpha \wedge E(r\alpha) - kH(r\alpha) = o(\frac{1}{r}) \,, \tag{1.4}$$

where

$$\begin{cases} r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}, \qquad k = k_0 = \omega(\epsilon_0\mu_0)^{\frac{1}{2}} = \frac{\omega}{c}, \\ \vec{\alpha} = \frac{\vec{x}}{|x|} = \frac{\vec{x}}{r}, \qquad \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

Let us first recall why system (1.3) along with the condition at infinity (1.4) has an unique solution *E*, *B*. For this purpose, we start by recalling some known facts for Helmholtz equation in $\mathbb{R}^3$, see the references [9,10,11, 12] for complete details.

Let the function $\Phi = \Phi^{out}$ defined on $\mathbb{R}^3$ by

$$\Phi(x) = \Phi(r) = \frac{e^{ikr}}{4\pi r} \tag{1.6}$$

It satisfies

$$\begin{cases} (\Delta + k^2)\Phi = -\delta_0, \\ \frac{\partial\Phi}{\partial r} - ik\Phi = -\frac{e^{ikr}}{4\pi r^2} = o(\frac{1}{r}). \end{cases} \tag{1.7}$$

$\Phi = \Phi^{out}$ corresponds to an outgoing wave due to the (point) source δ_0. Its conjugate is the elementary incoming solution $\Phi = \Phi^{\text{int}}$.

Furthermore, if f be a distribution in $\mathbb{R}^3$ with compact support, then the convolution product (which is an outgoing wave)

$$\vec{u} = \Phi * f, \tag{1.8}$$

satisfies the Helmholtz equation (1.2).

Unicity of Helmholtz equation together with the outgoing condition at infinity results from [19], [21]. With these preliminaries, we can eliminate in system (1.3) electric field *E* or the magnetic field *H*, in order to obtain equations on *E* or on *H* only. By standard computations, letting

$$\vec{m} = i\omega\mathrm{rot}(P\delta_0) = \mathrm{rot}J,$$
$$\vec{j} = i\omega\mu_0(J + k^{-2}\mathrm{grad}(\mathrm{div}J)), \tag{1.9}$$

we obtain the (vector) Helmholtz equations in $\mathbb{R}^3$

$$\begin{cases} \Delta H(x) + k^2 H(x) = -\vec{m}, \\ \Delta E(x) + k^2 E(x) = -\vec{j}, \end{cases} \tag{1.10}$$

where $\vec{m}, \vec{j}$ are distributions with compact support, defined by (1.9).

As in the case of scalar Helmholtz equation, it is necessary to introduce conditions at infinity in order to obtain a well-posed problem. Thus, we shall assume that system (1.10) satisfy the Silver-Muller condition at infinity (1.4).

Then system (1.10) with (1.4) has an unique solution (*E*, *H*) in $D'_c(\mathbb{R}^3)^3 \times D'_c(\mathbb{R}^3)^3$, see [12], given by the convolution products as

$$\vec{E} = \Phi * \vec{j}, \ \vec{H} = \Phi * \vec{m} \tag{1.11}$$

where $\Phi = \Phi^{out}$ is the elementary outgoing solution of the Helmholtz equation, given by (1.6).

In the following, we are interested in the behavior of these fields near a fixed sphere around the origin, assumed to be our single dipole.

Next, fix a radius $r_\varepsilon > 0$. Here $\varepsilon > 0$ is a small parameter and r_ε depends on ε. In fact ε will denote the typical distance between two dipoles (see figure 3 below at the end of paper).

From the above results, one obtains that the electric field and the magnetic field are linked on the surface S_{r_ε} of the ball of radius r_ε centered at the dipole by

$$\alpha \wedge E = H(-\frac{k^2}{i\varepsilon_0\omega})(\frac{r_\varepsilon}{ikr_\varepsilon - 1} + \frac{1}{k^2 r_\varepsilon}) \ \text{on } S_{r_\varepsilon} \tag{1.12}$$

Denoting by $Z_0 = \frac{k}{\varepsilon_0\omega}$ the impedance of the vacuum, see [12], we have also

$$\alpha \wedge E = -\zeta_\varepsilon Z_0 H, \tag{1.13}$$

where $\zeta_\varepsilon = \frac{kr_\varepsilon}{kr_\varepsilon + i} + \frac{i}{kr_\varepsilon}$. Note also that $\alpha.H = 0$.

Denoting by C^ε_{inc} the local (scattering) Calderon operator, giving the relation between the tangential components of *E* and *H*, and letting $\Pi_{S_{r_\varepsilon}}$ be the projection on the tangent plane to the sphere S_{r_ε} then from (1.13), one can write

$$C^\varepsilon_{inc}(\alpha \wedge E) = \alpha \wedge H = \frac{1}{\zeta_\varepsilon Z_0}\alpha \wedge (\alpha \wedge E) = -\frac{1}{\zeta_\varepsilon Z_0}\Pi_{S_{r_\varepsilon}} E \tag{1.14}$$

Since $E = (E.\alpha)\alpha + \Pi_{S_{r_\varepsilon}} E$, it follows that $E = (E.\alpha)\alpha + \Pi_{S_{r_\varepsilon}} E$. It is important to notice that relation (1.14) is independent from dipolar momentum and position of the (single) particle. We remark, see [12], that

$$\Re(\zeta_\varepsilon) == \frac{k^2 r_\varepsilon^2}{k^2 r_\varepsilon^2 + 1} \geq 0 \tag{1.15}$$

and thus

$$\Re(\zeta_\varepsilon) \leq \frac{1}{r_\varepsilon}$$

Notice that as $r_\varepsilon \to 0$ we have, $\zeta_\varepsilon = O(\frac{i}{r_\varepsilon})$.

Furthermore, the Calderon operator C^ε_{inc} satisfies on the boundary of our domain ($\partial\Omega = \Gamma$) the relation (similarly to the exterior Calderon operator)

$$\begin{cases} \Re(\int_\Gamma (\alpha \wedge H).\bar{E}d\Gamma) = \Re(\int_\Gamma C^\varepsilon_{inc}(\alpha \wedge E).\bar{E}d\Gamma) \\ \qquad = -\Re(\zeta_\varepsilon)Z_0 \int_\Gamma |H|^2 d\Gamma \leq 0\,. \end{cases} \tag{1.16}$$

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III. Microscopic Problem For N Charged Particles

Now, we can consider an assembly of N particles inside our domain $\Omega \subset \mathbb{R}^3$ and apply the previous framework for this purpose.

Let (E_{inc}, H_{inc}) be the incident electromagnetic field with frequency ω and λ the wave length of the incident electromagnetic field. Recall that $\lambda = \frac{2\pi\omega}{c}$, where $c > 0$ is a constant.

Let $\varepsilon > 0$ be a small parameter identified as being the previous parameter ε. We assume our N particles are located inside a box Ω' strictly included in Ω, ($\text{dist}(\Omega', \partial\Omega) > 0$), at point εj, $j = 1......N$. Let $B^j_{r_\varepsilon}$ be the ball with radius $r_\varepsilon k = \frac{\lambda}{2\pi} r_\varepsilon < \varepsilon/2$, centered at x_j.

This assumption seems to be superfluous, but in fact it is linked with the aim of getting uniform a priori estimates near the boundary.

We denote by Ω_ε the domain Ω with all the balls $B^j_{r_\varepsilon}$ removed. We note by $S^j_{r_\varepsilon}$, $j = 1......N$ the surfaces of these balls. In view of the previous section, let $(E^\varepsilon, H^\varepsilon)$ be the electromagnetic field in Ω_ε, which is created by the incident field. It is then solution of Maxwell's problem

$$\begin{cases} H^\varepsilon + i\omega\varepsilon_0 E^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon, \\ -\text{rot}E^\varepsilon + i\omega\mu_0 H^\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon, \\ n \wedge H^\varepsilon - C^e(n \wedge E^\varepsilon) = f_{inc} & \text{on} \quad \Gamma = \partial\Omega, \\ n \wedge H^\varepsilon - C^\varepsilon_{inc}(n \wedge E^\varepsilon) = 0 & \text{on} \quad S^j_{r_\varepsilon} = \partial B^j_{r_\varepsilon} \end{cases} \tag{1.17}$$

where C^ε_{inc} is given by (1.14) and

$$f_{inc} = n \wedge H_{inc} - C^e(n \wedge E_{inc}) \tag{1.18}$$

with E_{inc} given as above and C^e is the exterior Calderon operator which describes the scattering from the exterior of Ω. We describe more precisely its action in the next section. Problem (1.17) is the interior problem.

3.1 The exterior problem

In this section, we describe the exterior problem arising in our electromagnetic problem.

Let Ω be a bounded connected domain. Let us first describe the exterior Calderon operator C^e We will forget the parameter ε without possible confusion. The exterior Calderon operator C^e is defined as the mapping

$$C^e \; : \; H^{-1/2}(\text{div}, \partial\Omega) \to H^{-1/2}(\text{div}, \partial\Omega)$$

given by $C^e(m) = \vec{n} \wedge H_1$, where (E_1, H_1) is solution of the following exterior problem (with $m = -n \wedge E_{inc}$, for some E_{inc})

$$(Ext) \quad \begin{cases} \text{rot}H_1 + i\omega\epsilon_0 E_1 = 0 \;\text{ in }\; \Omega^c, \\ -\text{rot}E_1 + i\omega\mu_0 H_1 = 0 \;\text{ in }\; \Omega^c, \\ \omega\epsilon_0\alpha \wedge E_1(r\alpha) - kH_1(r\alpha) = o(\frac{1}{r}) \;\text{ as }\; r \longrightarrow \infty, \\ n \wedge E_1 = m, \;\text{ on }\; \Gamma = \partial\Omega, \\ (E_1, H_1) \in H_{\text{loc}}(\text{rot}, \bar{\Omega}^c) \times H_{\text{loc}}(\text{rot}, \bar{\Omega}^c), \end{cases}$$

Recall that, see [12]

$$H_{\text{loc}}(\text{rot}, \overline{\Omega}^c) = \{u \in D'(\overline{\Omega}^c), \xi u \in H(\text{div}, \overline{\Omega}^c), \forall \xi \in D(\mathbb{R}^3)\}$$

and the trace space is defined as

$$H^{-1/2}(\text{div}, \partial\Omega) := \{u \in H^{-1/2}(\partial\Omega)^3, \vec{n}.\vec{u} = 0, \text{div}_{\partial\Omega} u \in H^{-1/2}(\partial\Omega)\}.$$

Proposition 1. For a given $m = -n \wedge E_{inc}$ in $H^{-1/2}(div, \partial\Omega)$, the exterior problem (Ext) has a unique solution in $H_{\text{loc}}(\text{rot}, \overline{\Omega}^c) \times H_{\text{loc}}(\text{rot}, \overline{\Omega}^c)$.

The **proof of Proposition Ext** follows from [12, 22, 19].

IV. Variational Framework and Uniform Estimates

A. Uniform estimates for the interior problem

From problem (1.17), we obtain, for all F in $H(\text{rot}, \Omega_\varepsilon) = \{u \in L^2(\Omega_\varepsilon)^3, \text{ rot}u \in L^2(\Omega_\varepsilon)^3\}$, the following (weak) formulation

$$\begin{cases} -\frac{1}{i\omega\mu_0}\int_{\Omega_\varepsilon} \text{rot}E^\varepsilon.\text{rot}\bar{F}\, dx - i\omega\epsilon_0 \int_{\Omega_\varepsilon} E^\varepsilon.\bar{F}\, dx \\ -\int_\Gamma C^e(n \wedge E^\varepsilon).\bar{F}\, d\Gamma - \sum_{j=1}^N \int_{S^j_{r_\varepsilon}} C^\varepsilon_{inc}(n \wedge E^\varepsilon).\bar{F}\, dS \;=\; \int_\Gamma f_{inc}.\bar{F}\, d\Gamma. \end{cases} \tag{2.19}$$

Let $\tilde{C}^e \equiv i\omega\mu_0 C^e$ and Since $K = \frac{\omega\mu_0}{Z_0}$, μ_0, are constants in our domain, we have the following variational formulation

$$\begin{cases} \int_{\Omega_\varepsilon} \text{rot}E^\varepsilon.\text{rot}\bar{F}\, dx - k^2 \int_{\Omega_\varepsilon} E^\varepsilon.\bar{F}\, dx \\ +\int_\Gamma \tilde{C}^e(n \wedge E^\varepsilon).\bar{F}\, d\Gamma + +iK\zeta^{-1}\sum_{j=1}^N \int_{S^j_{r_\varepsilon}} (\Pi_S E^\varepsilon).\bar{F}\, dS = -i\omega\mu_0 \int_\Gamma f_{inc}.\bar{F}\, d\Gamma. \end{cases} \tag{2.20}$$

This is the natural variational formulation associated with problem (1.17), (1.18). Let us set

$$\begin{cases} a^\varepsilon(E^\varepsilon, F) \equiv -\frac{1}{i\omega\mu_0}\int_{\Omega_\epsilon} \text{rot}E^\varepsilon.\text{rot}\bar{F}\, dx - i\omega\epsilon_0 \int_{\Omega_\varepsilon} E^\varepsilon.\bar{F} \\ -\int_\Gamma C^e(n \wedge E^\varepsilon).\bar{F}\, d\Gamma - \sum_{j=1}^N \int_{S^j_{r_\varepsilon}} C^\varepsilon_{inc}(n \wedge E^\varepsilon).\bar{F}\, dS \end{cases} \tag{2.21}$$

Recall that, see [12], the variational formulation a^ε is clearly a sesquilinear and continuous, and also coercive, there is a constant α_0 such that

$$\Re\{a^\varepsilon(E^\varepsilon, E^\varepsilon)\} \geq \alpha_0 ||E^\varepsilon||^2_{H(\text{rot},\Omega_\varepsilon)}. \tag{2.22}$$

The form $F \to \int_\Gamma f_{inc}.\bar{F}\, d\Gamma$ being continuous on $H(\text{rot}, \Omega_\varepsilon)$, we can apply, Lax-Milgram Lemma, and thus

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problem (1.17), (1.18) has an unique solution in $H(\text{rot},\Omega_\varepsilon)$ under the following variational formulation: $E^\varepsilon \in H(\text{rot},\Omega_\varepsilon)$ satisfies

$$a^\varepsilon(E^\varepsilon,F) = -i\omega\mu\int_\Gamma f_{inc}.\overline{F}\, d\Gamma,\ \forall F \in H(\text{rot},\Omega_\varepsilon) \quad (2.23)$$

For the uniform estimates (wrt ε), and by taking $\overline{F} = E^\varepsilon$ in (2.19), and using the uniform estimate for the exterior problem and Cauchy-Schwarz inequality, see [12], we have

$$\alpha \| E^\varepsilon \|^2_{H(rot,\Omega_\varepsilon)} \le | a^\varepsilon(E^\varepsilon, E^\varepsilon) | \le c \| E^\varepsilon \|_{H(\text{rot},\Omega_\varepsilon)}$$

Thus one has the uniform estimate

$$\| E^\varepsilon \|_{H(rot,\Omega_\varepsilon)} \le c.$$

B. Uniform estimate for the exterior problem

The exterior problem is given by the fields $(E_1^\varepsilon, H_1^\varepsilon)$ solutions of

$$\begin{cases} \text{rot}H_1^\varepsilon + i\omega\varepsilon_0 E_1^\varepsilon = 0 \text{ in } \Omega^c, \\ -\text{rot}E_1^\varepsilon + i\omega\mu_0 H_1^\varepsilon = 0 \text{ in } \Omega^c, \\ \omega\varepsilon_0\alpha \wedge E_1(r\alpha) - kH_1(r\alpha) = o(\tfrac{1}{r}) \text{ as } r \to \infty, \\ n \wedge E_1^\varepsilon = n \wedge (E^\varepsilon - E_{inc}), \quad \text{on } \Gamma = \partial\Omega, \\ (E_1^\varepsilon, H_1^\varepsilon) \in H_{\text{loc}}(\text{rot}, \overline{\Omega}^c) \times H_{\text{loc}}(\text{rot}, \overline{\Omega}^c). \end{cases}$$

Here E^ε is the transmitted electric field of the previous section and E_{inc} is given in $H^{-1/2}(\text{div},\partial\Omega)$. Using the uniform estimate of (1.17),(1.18) problem, one has, see [12]

$$\alpha \| E_1^\varepsilon \|^2_{H(rot,\overline{\Omega}^c)} \le | a^\varepsilon(E_1^\varepsilon, E_1^\varepsilon) | \le c \| E_1^\varepsilon \|_{H(\text{rot},\overline{\Omega}^c)}$$

Thus one has

$$\| E_1^\varepsilon \|_{H(rot,\overline{\Omega}^c)} \le c$$

where c is a constant independent from ε.

V. Homogenization

In this paper, we shall study the above problem under one the following scaling assumptions

$$\lim_{\varepsilon\to 0} \frac{kr_\varepsilon}{\varepsilon} \qquad (HYP)_1$$

or

$$kr_\varepsilon = c\varepsilon \qquad (HYP)_2$$

where $0 < c < \frac{1}{4}$ is strictly positive fixed constant. We will proof the following theorem.

Theorem 1: Let E^ε be the variational solution of problem (1.17). Denoting by $\sim$ the extension by o in the holes, then (up to a subsequence)

$$\tilde{E}^\varepsilon \rightharpoonup E \text{ weakly in } L^2(\Omega)^3 \text{ and } \tilde{H}^\varepsilon \rightharpoonup H \text{ weakly in } L^2(\Omega)^3$$

Furthermore, we have

1) Under assumption $(HYP)_2$ and with (1.18), E is the variational solution of problem

$$\int_\Omega M[rot_x E(x)].rot\psi(x)dx - k^2\int_\Omega E(x).\psi(x)dx = -i\omega\mu_0\int_\Gamma f_{inc}.\psi(x)d\Gamma,\ \ \forall\psi \in H(rot,\Omega)$$

where $M \equiv \int_{Y^*} \chi_{Y^*}(y)\mathbb{G}(y)\times\mathbb{G}(y)dy$ is a constant (given below) and χ_{Y^*} is the characteristic function of χ_{Y^*}.

2) Under assumption $(HYP)_1$ and with (1.18), E is the variational solution of problem

$$-\frac{1}{i\omega\mu_0}\int_\Omega \text{rot}E.\text{rot}\overline{\psi}(x)\,dx - i\omega\varepsilon_0\int_\Omega E.\overline{\psi}(x)\,dx = \int_\Gamma f_{inc}.\overline{\psi}(x)d\Gamma.$$

for all $\psi \in H(rot,\Omega)$.

The above (mathematical) formulations (in case 1) are expressing the fact that the weak limits satisfy a modified Maxwell's equation, with effective electric constants inside Ω, that is more precisely

$$\begin{cases} \text{rot}H - i\omega\varepsilon_0 E = 0, \text{ in } \Omega, \\ \text{rot}E - i\omega\mu_0 M^{-1} H = 0 \text{ in } \Omega \end{cases}$$

Taking into account the exterior problem, it shows that the global frequency has changed, since the global frequency in our domain depends on the length of each particle and on the boundary conditions.

Proof of theorem 1.

1) Under assumption $(HYP)_2$

By (2.24), $(\tilde{E}^\varepsilon)$ is bounded in $L^2(\Omega)^3$, as well as $(\tilde{\text{r}}\text{ot}E^\varepsilon)^3$. Therefore, we can extract two sub-sequences such that

$$\tilde{E}^\varepsilon \rightharpoonup E = E(x) \text{ in } (L^2(\Omega))^3 \text{ weakly}, \qquad (3.25)$$

$$\tilde{rot}\, E^\varepsilon \rightharpoonup V = V(x) \text{ in } (L^2(\Omega))^3 \text{ weakly} \qquad (3.26)$$

From [2, 17, 22], we can also arrange that for the same sequences, there exists two two-scale limits, $E^0 \in L^2(\Omega\times Y)^3, V^0 \in L^2(\Omega\times Y)^3$, such that for any $\psi(x,y) \in D(\Omega, C_p^\infty(Y))^3$, one has

$$\lim_{\varepsilon\to 0}\int_\Omega \tilde{E}^\varepsilon.\psi(x,\frac{x}{\varepsilon})dx = \int_\Omega\int_{Y^*} E^0(x,y).\psi(x,y)dxdy,\ \int_{Y^*} E^0(x,y)dy = E(x) \qquad (3.27)$$

$$\lim_{\varepsilon\to 0}\int_\Omega \text{rot}\tilde{E}^\varepsilon.\psi(x,\frac{x}{\varepsilon})dx = \int_\Omega\int_{Y^*} V(x,y).\psi(x,y)dxdy,\ \int_{Y^*} V^0(x,y)dy = V(x) \qquad (3.28)$$

For all $\psi(x,y) \in D(\Omega, C_p^\infty(Y^*))^3$ with compact support (in y) in Y^*, let us note that an integration by parts yields

$$\varepsilon\int_\Omega \tilde{\text{r}}\text{ot}E^\varepsilon.\psi(x,\frac{x}{\varepsilon})dx = \int_\Omega \tilde{E}^\varepsilon.[\varepsilon\text{rot}_x\psi(x,\frac{x}{\varepsilon}) + \text{rot}_y\psi(x,\frac{x}{\varepsilon})]dx.$$

Passing to the limit in both terms with the help of two-scale convergence gives

$$\int_\Omega\int_{Y^*} E^0(x,y).\text{rot}_y\psi(x,y)dxdy = 0$$

that is

$$\text{rot}_y E^0(x,y) = 0 \text{ in } D'(\Omega\times Y^*)^3 \qquad (3.29)$$

Now, we add to the assumption on $\psi(x,y)$ the condition $\text{rot}_y\psi(x,y) = 0$, with compact support in x as well as in y in Y^*. Since

$$\int_\Omega \tilde{\text{r}}\text{ot}E^\varepsilon(x).\psi(x,\frac{x}{\varepsilon})dx = \int_\Omega \tilde{E}^\varepsilon(x).\text{rot}_x\psi(x,\frac{x}{\varepsilon})dx,$$

using two-scale convergence, we are led this time to

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$$\int_\Omega\int_{Y^*} V(x,y).\psi(x,y)dxdy = \int_\Omega\int_{Y^*} \tilde{E}^0(x,y).\mathrm{rot}_x\psi(x,y)dxdy \tag{3.30}$$

and thus integrating by parts in Ω, we get

$$\int_\Omega\int_{Y^*} E^0(x,y).\mathrm{rot}_x\psi(x,y)dxdy = \int_\Omega\int_{Y^*} \mathrm{rot}_x E^0(x,y).\psi(x,y)dxdy$$

Hence

$$\int_\Omega\int_{Y^*}[V(x,y) - \mathrm{rot}_x E^0(x,y)].\psi(x,y)dxdy = 0 \tag{3.31}$$

We shall use all these facts to identify E^0 and V.

As $\mathrm{rot}_y E^0(x,y) = 0$, then there exist $\phi_1 \in H^1(Y^*)$ $h_1 \in H_1(Y^*)$ such that, see [3, 4, 5, 12]

$$E^0(x,y) = \mathrm{grad}_y\phi_1(x,y) + h_1(x,y) \tag{3.32}$$

where

$$H_1(Y^*) = \{u \in L^2(Y^*)^3, \mathrm{rot}_y u = 0, \mathrm{div}_y u = 0, \vec{n}(y).\vec{u}(y) = 0\}$$

and one can write (here χ_{Y^*} being the characteristic function of Y^*)

$$E^0(x,y) = \chi_{Y^*}(y)\mathbb{G}(y)[E(x)] \tag{3.33}$$

Here $\mathbb{G}(y)$ is the $\prod_3$ valued function defined, for $y \in Y^*$ by

$$\mathbb{G}(y) = \begin{pmatrix} \vec{G}_1(y) & \vec{G}_2(y) & \vec{G}_3(y) \end{pmatrix}, \ \vec{G}_i(y) = (\nabla\tilde{\Phi}_i)(y) \tag{3.34}$$

each $\tilde{\Phi}_i$ being solution in $H^1(Y^*)$ of

$$\begin{cases} \tilde{\Phi}_i = y_i - \Upsilon_i, \ \Upsilon_i \ Y - \text{periodic}, \\ \quad \Delta\tilde{\Phi}_i = 0, \\ \dfrac{\partial\tilde{\Phi}_i}{\partial n}|_{\partial T} = n.\nabla\tilde{\Phi}_i = 0. \end{cases} \tag{3.35}$$

Note that $\mathrm{rot}_y\mathbb{G} = 0$ and $\vec{n}(y)\cdot\mathbb{G}(y) = 0$. We use (3.31) and (3.33) to get

$$\int_\Omega\int_{Y^*} V^0(x,y).\psi(x,y)dxdy = \int_\Omega\int_{Y^*} \chi_{Y^*}(y)\mathbb{G}(y)[E(x)].\mathrm{rot}_x\psi(x,y)dxdy$$

Integrating by parts in Ω, one has

$$\int_\Omega\int_{Y^*}[V^0(x,y) - \mathrm{rot}_x(\chi_{Y^*}(y)\mathbb{G}(y)[E(x)])].\psi(x,y)dxdy = 0$$

for any $\psi(x,y) \in \psi(x,y) \in D(\Omega, C_p^\infty(Y))^3$. Thus

$$V^0(x,y) = \mathrm{rot}_x(\chi_{Y^*}(y)\mathbb{G}(y)[E(x)]) \ \text{ in } \ D'(\Omega, C_p^\infty(Y))^3 \tag{3.36}$$

Thus one has, see [3,4,5,10]

$$V(x,y) = \chi_{Y^*}(y)\mathbb{G}(y) \times \mathbb{G}(y)[\mathrm{rot}_x E(x)]. \tag{3.37}$$

Now, consider the oscillating test function $\mathbb{W}^\varepsilon$ constructed in [1] but in the case $kr^\varepsilon = c\,\varepsilon$ with Ω' as domain. In Ω', one has

$$\mathbb{W}^\varepsilon = \mathbb{W}(\frac{x}{\varepsilon})$$

where $\mathbb{W}$ is a fixed periodic function of y, see again [1] for details. Of course

$$\mathbb{W}^\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y \mathbb{W}(y)dy = aId \ \text{ weakly in } \ (L^2(\Omega'))^3$$

where a is a constant. We extend $\mathbb{W}$ by identity outside of Ω', that is in Ω/Ω'.

Now, for any $\psi \in C^\infty(\Omega)$, we take $\mathrm{W}^\varepsilon(\Omega)$ in the variational formulation (3.19).

Since $n \wedge \mathrm{W}^\varepsilon = 0$ on the small balls, it follows that

$$\int_\Omega \tilde{\mathrm{rot}}E^\varepsilon.\mathrm{rot}\{\mathbb{W}^\varepsilon[\psi](x)\}dx - k^2\int_\Omega \tilde{E}^\varepsilon.\mathbb{W}^\varepsilon[\psi](x)dx = -i\omega\mu_0\int_\Gamma f_{inc}.\mathbb{W}^\varepsilon[\psi](x)\ d\Gamma \tag{3.38}$$

As $\varepsilon \to 0$, one has

$$\int_\Omega\int_{Y^*} V(x,y).\mathrm{rot}\psi(x)dxdy - k^2\int_\Omega E(x).\psi(x)dx = -i\omega\mu_0\int_\Gamma f_{inc}\psi(x)d\Gamma, \tag{3.39}$$

which, on using (3. 37), yields

$$\int_\Omega M[\mathrm{rot}_x E(x)].\mathrm{rot}\psi(x)dx - k^2\int_\Omega E(x).\psi(x)dx = -i\omega\mu_0\int_\Gamma f_{inc}\psi(x)d\Gamma. \tag{3.40}$$

where $M \equiv \int_{Y^*}\chi_{Y^*}(y)\mathbb{G}(y) \times \mathbb{G}(y)dy$ is a constant.

For the exterior problem with the same assumptions, we find easily that the limit of the electric field E_1^ε (resp. the magnetic field E_1^ε) satisfies a homogenized system with the same constant coefficients. This concludes the proof of *Theorem 1:* In case of assumption $(HYP)_2$.

2) Under assumption $(HYP)_1$

Again, using (2.24), we can still assume that

$$\tilde{E}^\varepsilon(x) \rightharpoonup E(x) \quad \text{weakly in } \ L^2(\Omega), \tag{3.41}$$

$$\mathrm{rot}\tilde{E}^\varepsilon \rightharpoonup V(x) \quad \text{weakly in } \ L^2(\Omega)^3. \tag{3.42}$$

For all ψ in $C^\infty(\overline{\Omega})^3$, one has

$$\begin{cases} -\dfrac{1}{i\omega\mu_0}\displaystyle\int_{\Omega_\varepsilon}\mathrm{rot}E^\varepsilon.\mathrm{rot}\bar{\psi}\,dx - i\omega\epsilon_0\int_{\Omega_\varepsilon}E^\varepsilon.\bar{\psi}\,dx \\ -\displaystyle\int_\Gamma C^e(n\wedge E^\varepsilon).\bar{\psi}\,d\Gamma - \sum_{j=1}^N\int_{S^j_{r_\varepsilon}} C^\epsilon_{inc}(n\wedge E^\varepsilon).\bar{\psi}\,dS = \int_\Gamma f_{inc}.\bar{\psi}\,d\Gamma. \end{cases} \tag{3.43}$$

Firstly let us show that

$$\mathrm{rot}\tilde{E}^\varepsilon \rightharpoonup V \equiv \mathrm{rot}_x E \quad \text{weakly in } \ L^2(\Omega)^3. \tag{3.44}$$

We use the oscillating test function W^ε given in [1] but in Ω' and extended by identity in Ω/Ω'.

Let $\vec{\varphi} \in (C_c^\infty(\Omega))^3$. Then

$$\begin{cases} \displaystyle\int_\Omega \mathrm{rot}\tilde{E}^\varepsilon.\vec{\varphi} = \int_{\Omega_\varepsilon}\mathrm{rot}E^\varepsilon.\vec{\varphi} = \int_{\Omega_\varepsilon}\mathrm{rot}E^\varepsilon.\mathbb{W}^\varepsilon[\vec{\varphi}] + \int_{\Omega_\varepsilon}\mathrm{rot}E^\varepsilon.\{\vec{\varphi} - \mathbb{W}^\varepsilon[\vec{\varphi}]\} \\ \qquad \longrightarrow \displaystyle\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\mathrm{rot}E^\varepsilon.\mathbb{W}^\varepsilon[\vec{\varphi}] \end{cases} \tag{3.45}$$

since the $\varphi - \mathrm{W}^\varepsilon[\varphi] \to 0$ strongly in $L^2(\Omega)^3$. Then

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon}\mathrm{rot}E^\varepsilon.\mathbb{W}^\varepsilon[\vec{\varphi}] = \int_{\Omega_\varepsilon}E^\varepsilon.\mathrm{rot}_x\{-\mathbb{W}^\varepsilon[\vec{\varphi}]\} \\ \qquad = \displaystyle\int_{\Omega_\varepsilon}E^\varepsilon.\mathbb{W}^\varepsilon \wedge \nabla\vec{\varphi} \longrightarrow \int_\Omega E.\mathrm{rot}_x\vec{\varphi} \end{cases} \tag{3.46}$$

Thus

$$V = \mathrm{rot}_x E$$

Now, we take in (4.19) ψ under the form $\mathrm{W}^\varepsilon[\psi]$, $\forall\vec{\psi} \in (C^\infty(\Omega))^3$ and we have

$$-\frac{1}{i\omega\mu_0}\int_\Omega \mathrm{rot}E.\mathrm{rot}\vec{\psi}\,dx - i\omega\epsilon_0\int_\Omega E.\vec{\psi}\,dx = \int_\Gamma f_{inc}.\vec{\psi}d\Gamma$$

VI. Conclusion

We have fined the homogenized equations for our problem, interior problem and exterior problem, and we gets the same homogeneous constant which is normal because the two problems are lies, but as on our boundary we have two kinds of waves, reflected and transmitted, thus we gets two different effective frequency.

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References

[1] ALEXANDRE, R., TAHA, H. *Propagation of electromagnetic waves in non-homogeneous media* . Applications of Mathematics. No. 3, 201-225, (2004).

[2] ALLAIRE, G. Homogenization and two-scale convergence. *SIAM. J. Math. Anal.* 23(1992),1482-2518.

[3] ARTOLA, M., CESSENAT, M. *Sur un probleme raide en homogeneisation elliptique.C.R.Acad. Sci.Paris*, t.309, S´erie I (1989), 761-766.

[4] ARTOLA, M., CESSENAT, M. *Un probleme raide avec homogeneisation en electromagnetisme.C.R.Acad. Sci.Paris*, t. 310,S´erie I (1990), 9-14.

[5] ARTOLA, M., CESSENAT, M. *Sur la propagation des ondes electromagnetiques dans un milieu composite (dielectrique-conducteur). C.R.Acad. Sci.Paris*, t.310,Serie I (1990), 375-380.

[6] ARTOLA, M., CESSENAT, M. *Problemes d'homogeneisation dielectrique conducteur dans les composites en structure periodique.* Rapport CEA-DAM, Limeil-Valenton (1990).

[7] BENSOUSSAN, A., LIONS, J.-L., PAPANICOLAOU, G. *Asymptotic analysis for periodic structures*. North-Holland (1978).

[8] BORN, M., WOLF, E. *Electromagnetic Theory of propagation, interference and diffraction of light*. New york (1975).

[9] BOSSAVIT, A. *Electromagnetisme en vue de la modelisation*. Springer-Verlag, Paris (1991).

[10] BOUCHE, D., MOLINET, F. *Methodes Asymptotiques en electromagnetisme*. New York (1994).

[11] BOUIX, M. *Les discontinuites du rayonnement electromagnetique*. Dunod, Paris (1965).

[12] CESSENAT, M. *Mathematical Methods In Electromagnetism Linear Theory and Applications*. World Scientific Publishing Co, Inc, River Edge, NJ (1996).

[13] CIORANESCU, D., MURAT, F. *Un terme etrange venu d'ailleurs. Nonlinear partial differential equations and their applications*, College de France seminar, Volume II, ed. By H.Brezis and J.L.Lions, Research Notes in Mathematics, Pitman, London (1982), 98-138.

[14] CIORANESCU, D., MURAT, F. *Un terme etrange venu d'ailleurs II. Nonlinear partial differential equations and their applications*, College de France seminar, Volume II, ed. By H.Brezis and J.L.Lions, Research Notes in Mathematics, Pitman, London (1982), 154-178.

[15] CIORANESCU, D., DONATO, P. *An introduction to homogenization*. Oxford University Press, New York Paris (1999).

[16] DAUTRAY, R, LIONS, J.-L. *Analyse Mathematique et Calcul Numerique pour les Sciences et les Techniques*. Masson (1985), Vol 2.

[17] NGUETSENG, G. *A general convergence result for a functional related to the theory of homogenization*. SIAM J.Math.Anal, Vol.20 (3) (1989), 608-623.

[18] HUBERT, J.-S., SANCHEZ-PALANCIA, E. *Introduction aux Methodes asymptotiques et a l'homogeneisation*. Application `a la mecanique des milieux continus, Masson (1992).

[19] SANCHEZ-PALANCIA, E. *Non-homogeneous media and vibration theory*. Lectures Notes in Physics 127.Springer-Verlag, Berlin-Heidelberg-New York (1980).

[20] TARTAR, L. *Cours Peccot*. Unpublished College de France (1977).

[21] TARTAR, L. *Compensated compactness and applications to partial differential equations*. Nonlinear analysis and mechanics, Vol.IV, Pitman, Boston, Mass-London (1979), 136-212.

[22] WELLANDER, N. *Homogenization of Maxwell Equations, case I. Linear Theory*. Applications of Mathematics, No.1-46 (2001), 29-51.

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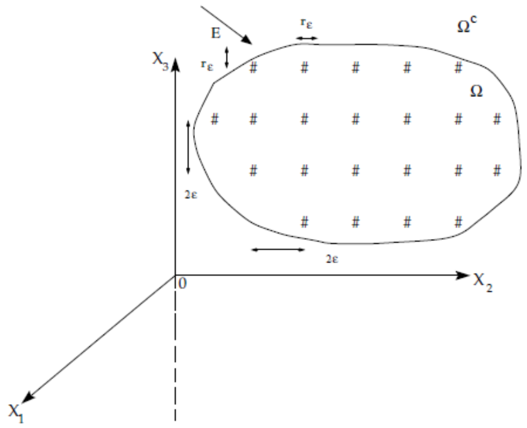

Fig.1. Ω a bounded regular open set of $\mathbb{R}^3$. In Ω , we consider a given assembly of N particles and outside Ω , an incident (on $\partial\Omega$) electromagnetic field with a fixed frequency $\omega > 0$

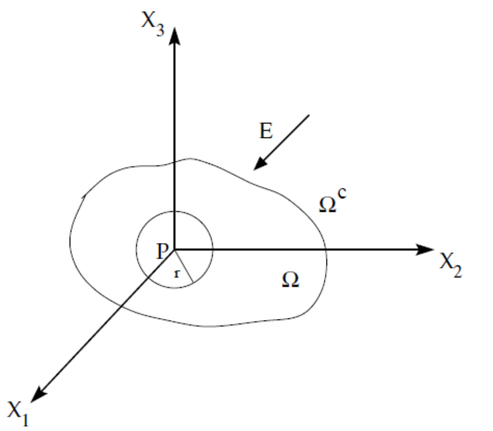

Fig.2. Problem of only one dipole, centered at the origin with a dipolar momentum P

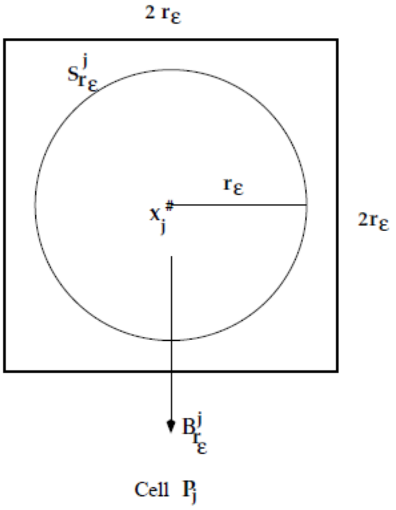

Fig.3. Distance between two dipoles.