International Journal of Innovation in Science and Mathematics
Volume 12, Issue 6, ISSN (Online): 2347–9051

Formulation of Averaged Lagrangian Velocities for the Surface Wave Motion in a Layered Fluid

Mst. Shahana Parvin* and Md. Ashrafuzzaman Khan
Professor, Department of Applied Mathematics University of Rajshahi, Rajshahi-6205, Bangladesh.
*Corresponding author email id: shahana_amath79@yahoo.com

Date of publication (dd/mm/yyyy): 14/12/2024

***Abstract* – The Navier-Stokes equations are the nonlinear partial differential equations describing fluid motion derived from the application of Newton's second law of fluid motion. To derive the solution of Navier-Stokes equations of motion which gives the notion of velocity field components with different densities [S. Parvin and S. Sultana (2024)] and its application on many branches of fluid dynamics. For the irrotational wave motion, the Lagrangian formulation is straight forward and follows from a Lagrangian functional which is the difference between the kinetic energy and the potential energy of the system. When the velocity decomposition consists of a potential part and a shear current in two different densities, the vorticity is constant. Lagrangian velocity is formulated with a shear current and without a shear current in both densities. The averaged Lagrangian velocities are also derived in both cases.**

***Keywords* – Navier-Stokes Equations of Motion, Lagrangian and Averaged Lagrangian Velocities, Shear Currents.**

I. Introduction

In many fields and laboratory studies and in engineering applications, the full Navier-Stokes equations appear complex situation for modeling at hand and consequently there have appeared many approximate models applying to restricted physical regimes. The Navier-Stokes equations are of great interest in a mathematical sense. The Navier-Stokes equations are a set of partial differential equations for describing the motion of viscous fluid substances according to French engineer and physicist Cloud-Louis Navier and Anglo-Irish physicist and mathematician George Gabriel Stokes. The resulting Navier-Stokes equations are very difficult to solve because it is nonlinear. Some simplifying assumption can be used for the solution of Navier-Stokes equations that provide velocity field with different properties and its application. In the 1870s, Boussinesq derived some model evolution equations which are applicable to describe for two dimensional motion and which have the form of a perturbation of the one dimensional wave equation. In fact, the solutions of Navier-Stokes equations have a great significance in natural sciences and it dominates a vast area of natural sciences such as mathematics, physics, mechanics, ordinary fluid dynamics, magneto-hydrodynamics, water waves, meteorology, plasma physics, engineering fields etc. Recently, many scientists generalized several methods to derive solution of Navier-Stokes equations and its application on many branches of fluid dynamics, such as series expansion method, finite difference method, exp-function method, Jacobi elliptic function method, variable separation method, numerical method, meshless local Pertrov-Galerkin method, variation method, meshless local radial basis function method etc. A huge amount of water waves at the free surface of a fluid of constant density can be focused for irrotational flow. The most general form of the Navier-Stokes equations is $\rho \frac{D\bar{q}}{dt} = -\nabla P + \nabla.\bar{T} + \bar{f}$, which cannot be applied until it has been more specified. Interior of the fluid, the Laplace equation must be determined by two nonlinear coupled equations at the free surface: one describing the evolution in time of the free surface itself, the other ensuring the boundary condition for the problem. Conservation of energy leads to Hamiltonian formulation and the relation between Hamiltonian and Lagrangian functions that can be experimentally detected in future. In the theory of surface waves, the difference between the mean velocity at a given point and the mass-transport velocity has long been recognized. Essentially, this difference is the same as the difference between the Eulerian mean and the Lagrangian mean velocity. The averaged Lagrangian approach applies to wave motion possibly superposed on a mean motion that can be described in a Lagrangian formulation. The Lagrangian is associated with the kinetic energy and potential energy of the motion; the oscillations contribute to the Lagrangian, although the mean value of the wave's oscillatory excursion is zero.

Choudhuri et. al. [1] derived a Lagrangian based approach to study the compatible Hamiltonian structure of the dispersionless KdV and supersymmetric KdV hierarchies and claim that their treatment of the problem serves as a very useful supplement of the so-called r-matrix method. Nick Pizzo [2] derived deep water surface gravity waves, and put in a form more suitable for numerical applications. M. Bjornestad et. al. [3] evaluated Lagrangian measurements of orbital velocities in the surf zone. While laboratory studies in shallow water confirmed some theoretical predictions about infragravity wave formation and shoaling (Lin & Hwung, [24]) as well as particle motion in gravity waves (Chen et al., [23]; Umeyama, [21]) and wave groups (Calvert et al., [22]), wave-by-wave properties of orbital velocities and mass transport remain largely unexplored in the field. A. Constantin [4] concerned with an alternative derivation of the Camassa-Holm equation for periodic waves, using a variational approach in the Lagrangian formalism. Hongwu Zhao and Kamran Mohseni [5] developed a dynamic procedure for the Lagrangian-averaged Navier-Stokes-α (LANS-α) equations where the variation in the parameter α in the direction of an isotropy determined in a self-consistent way from the data contained in the simulation itself. Kamran Mohseni [6] developed of Lagrangian averaging tools for analyzing flow over a lifting airfoil. Harish S. Bhat et. al [7] considered a 1-dimensional Lagrangian averaged model for an inviscid compressible fluid and presented a traveling wave analysis and a numerical study for such a model. Harish S. Bhat & C. Fetecau [8] also formulated the Lagrangian averaging for the 1-dimensional compressible Euler equations. Ambrosi [9] established Hamiltonian formulation for surface waves in a layered fluid with gravity force. Marche [10] studied the derivation with asymptotic analysis of two-dimensional viscous shallow water model in rotating framework with irregular topography, linear and quadratic bottom terms and capillary effects considering the three-dimensional Navier-Stokes equations with a free moving surface boundary condition and hydrostatic approximation. Longuet-Higgins [11] showed a simple physical model to obtain the Lagrangian characters including particle motion, mass transport, the Lagrangian wave period and the Lagrangian mean level for the surface waves that cannot directly obtain throughout the entire flow field. Perez and C. Ramirez [12] discussed Lagrangian formulation for permionic and super symmetric non-conservative systems. Parvin et al. [13] derived mass transport velocity using boundary conditions at water bottom and the flap surface and established the time average of horizontal velocity component at zero mean level. Parvin et al. [14] also formulated the wave frequency in terms of wave number depending on linearized term and also established phase speed with viscous term for long wave length. Sadeghi [15] derived Navier-Stokes equations from Lagrange’s equations and also showed the derivation of governing equations for viscous flow, the standard Lagrangian is more sufficient than non-standard Lagrangian. Didier Clamond [19] concerned the mathematical formulation of two-dimensional steady surface gravity waves in a Lagrangian description of motion. Jan Erik H. Weber [20] considered solutions for the Lagrangian mean drift in various types of surface waves influenced by viscosity.

Some aspects of fluid dynamics to derive the solution of Navier-Stokes equations of motion using variables s-

-eparation method, Parvin M.S. and Sultana M.S., [17] formulated velocity potentials for irrotational, incompressible two dimensional flow of two different densities with gravity force. For upper fluid density, velocity potential is trigonometric function and for lower fluid density, it contains hyperbolic function. Luc Deike et. al. [18] generally investigated the Lagrangian transport due to wave breaking surface waves. Conor Curtin & Rossen Ivanov [16] formulated the Lagrangian formulation for wave motion with a shear current and surface tension. In this paper, we use velocity potentials for irrotational, incompressible two dimensional flow of two different densities with gravity force, the Lagrangian and averaged Lagrangian velocities with a shear current and without a shear current in two different densities are formulated. Without a shear current, we have found that the aveaged Lagrangian velocity is zero in upper fluid and in lower fluid it is constant for parallel flow.

II. Mathematical Formulations

For the gravity force in a horizontally unbounded domain, we assume two fluids with constant densities ρ_1 and ρ_2. The velocity potentials of the lower fluid is denoted by $\varphi_1(x,z,t)$; $\infty \le z \le \tau_1$ and upper fluid, it is denoted by $\varphi_2(x,z,t)$; $\tau_1 \le z \le \tau_2$, where the surface $z=\tau_1(x,t)$ separates the two fluids and the surface $z=\tau_2(x,t)$ separates the upper fluid from the air. Consider the lower fluid is to be infinitely deep and the upper fluid of far field, the water is at rest, i.e., $\tau_1(\infty)=0$ *and* $\tau_2(\infty)=h=constant.$ The motion of the fluid is also governed by Laplace equation, i.e., velocity potentials $\varphi_1(x,y,z,t)$ and $\varphi_2(x,y,z,t)$ satisfy the Laplace equations $\Delta\ \varphi_1 + \varphi_{1ZZ} = 0$ and $\Delta\ \varphi_2 + \varphi_{2ZZ} = 0 = 0$, where $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplace operator acting on horizontal coordinates. For the separating surface, the velocity to fluids normal must be equal.

Naturally, various methods are applied to solve Navier-Stokes equations of motion. Using variables separation method, velocity vector fields are derived from Navier-Stokes equations of motion

$$\rho\left(\frac{\partial \bar{q}}{\partial t}+\left(\bar{q}.\nabla\right)\bar{q}\right)=\bar{F}-\nabla p+\mu\nabla^2\bar{q} \tag{1}$$

in Ω, where Ω be the domain which is occupied by viscous, incompressible fluid at time *t*. This system shows the Navier-Stokes equations of motion in each fluid for two cases, i.e., densities of upper and lower fluids and for incompressible fluid

$$\nabla.\bar{q}=0 \tag{2}$$

For small amplitude linear water waves, the Navier-Stokes equations of motion is linear. The potential energy is for upper fluid is

$$\varphi_1(x,z,t)=-\frac{a_n\omega}{k_n\sin(k_nh)}\cos k_n(z-h)\sin(\omega t)e^{-k_nx}. \tag{3}$$

where

$$K=\frac{\omega^2}{g}=k\tanh(kh)$$
$$=-k_n\tan(k_nh),\ n=1,2,\dots\dots\dots\dots \tag{4}$$

and for lower fluid,

$$\varphi_2\left(x,z,t\right)=-\frac{a_0\omega}{k\sinh\left(kh\right)}\cosh k\left(z-h\right)\cos\left(kx-\omega t\right) \tag{5}$$

In the Lagrangian reference frame, $\bar{\mathrm{X}}=(x,z)$, the Eulerian and Lagrangian reference frames connected by for a particle originally at $\bar{\mathrm{X}}_0=(x_0,z_0)$ are

$$\frac{d\bar{\mathrm{X}}}{dt}=\left[(\nabla\varphi)\right]\Big|_{(x=x_0,\,z=z_0)}$$

Therefore,

$$\frac{dx}{dt}=\frac{\partial\varphi}{\partial x}\Big|_{(x_o,z_0)}\ and\ \ \frac{dz}{dt}=\frac{\partial\varphi}{\partial z}\Big|_{(x_o,z_0)} \tag{6}$$

Hence for upper fluid of density ρ_1, the velocity components are

$$u_1=\frac{dx_1}{dt}=\frac{\partial\varphi_1}{\partial x}\Big|_{(x=x_0,z=z_0)}=(\varphi_1)_x\Big|_{(x=x_0,z=z_0)}=\frac{a_n\omega}{\sin\left(k_nh\right)}\cos k_n\left(z_0-h\right)\sin(\omega t)e^{-k_nx_0} \tag{7}$$

$$\text{and }\ w_1=\frac{dz_1}{dt}=\frac{\partial\varphi_1}{\partial z}\Big|_{(x=x_0,z=z_0)}=(\varphi_1)_z\Big|_{(x=x_0,z=z_0)}=\frac{a_n\omega}{\sin\left(k_nh\right)}\sin k_n\left(z_0-h\right)\sin(\omega t)e^{-k_nx_0} \tag{8}$$

and for lower fluid of density ρ_2, the velocity components are

$$u_2=\frac{dx_2}{dt}=\frac{\partial\varphi_2}{\partial x}\Big|_{(x=x_0,z=z_0)}=(\varphi_2)_x\Big|_{(x=x_0,z=z_0)}=\frac{a_0\omega}{\sinh\left(kh\right)}\cosh k\left(z_0-h\right)\sin\left(kx_0-\omega t\right) \tag{9}$$

$$w_2=\frac{dz_2}{dt}=\frac{\partial\varphi_2}{\partial z}\Big|_{(x=x_0,z=z_0)}=(\varphi_2)_z\Big|_{(x=x_0,z=z_0)}=-\frac{a_0\omega}{\sinh\left(kh\right)}\sinh k\left(z_0-h\right)\cos\left(kx_0-\omega t\right) \tag{10}$$

2.1. *The Velocity Field Decomposition Consists of a Potential Part and a Shear Current*

According to the Helmholtz theorem, if the velocity field decomposition consists of a potential part and a shear current, $\cup(z)=\gamma z+\kappa,$ which is linear, the vorticity $\nabla\times\bar{q}=\gamma$ is constant and κ is current constant at $z=0.$ Therefore, the velocity components in terms of potential energy φ and stream function ψ can be written as for upper fluid of density ρ_1,

$$\left.\begin{aligned}u_1&=(\varphi_1)_x+\cup_1(z)=(\varphi_1)_x+\gamma_1z+\kappa_1=(\psi_1)_z\\ w_1&=(\varphi_1)_z=-(\psi_1)_x\end{aligned}\right\} \tag{11}$$

and for lower fluid of density ρ_2,

$$\left.\begin{aligned}u_2&=(\varphi_2)_x+\cup_2(z)=(\varphi_2)_x+\gamma_2z+\kappa_2=(\psi_2)_z\\ w_2&=(\varphi_2)_z=-(\psi_2)_x\end{aligned}\right\} \tag{12}$$

(i) *Upper Fluid Density:*

The horizontal velocity component.

$$u_1 = (\varphi_1)_x + \cup_1(z) = (\varphi_1)_x + \gamma_1 z + \kappa_1$$

$$\therefore \frac{dx_1}{dt}\Big|_{(x=x_0, z=z_0)} = u_1 = \frac{a_n \omega}{\sin(k_n h)} \cos k_n (z_0 - h) \sin(\omega t) e^{-k_n x_0} + \gamma_1 z + \kappa_1$$

$$\therefore \int_{x_0}^{x} \frac{dx_1}{dt} dt = x_1 - x_{10} = -\frac{a_n}{\sin(k_n h)} \cos k_n (z_0 - h) \cos(\omega t) e^{-k_n x_0} + (\gamma_1 z_0 + \kappa_1) t$$

$$\therefore x_1 - x_{10} = -\frac{a_n}{\sin(k_n h)} \cos k_n (z_0 - h) \cos(\omega t) e^{-k_n x_0} + (\gamma_1 z_0 + \kappa_1) t. \tag{13}$$

Similarly, the vertical velocity component is

$$\frac{dz_1}{dt} = w_1 = \frac{a_n \omega}{\sin(k_n h)} \sin k_n (z - h) \sin(\omega t) e^{-k_n x} \Big|_{(x=x_0, z=z_0)}$$

$$\therefore \int_{z_0}^{z} \frac{dz_1}{dt} dt = z_1 - z_{10} = -\frac{a_n}{\sin(k_n h)} \sin k_n (z_0 - h) \cos(\omega t) e^{-k_n x_0}$$

$$\therefore z_1 - z_{10} = -\frac{a_n}{\sin(k_n h)} \sin k_n (z_0 - h) \cos(\omega t) e^{-k_n x_0} \tag{14}$$

The Lagrangian velocity at a point is found by Taylor expanding Eulerian velocity about (x_0, z_0).

Therefore,

$$U_{1L} = u_1(x_0, z_0, t) + (x_1 - x_{10}) \left(\frac{\partial u_1}{\partial x}\right)_{(x_0, z_0)} + (z_1 - z_{10}) \left(\frac{\partial u_1}{\partial z}\right)_{x_0, z_0,} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Applying various values in above equations and neglecting higher order terms, we get

$$U_{1L} = \frac{a_n \omega}{\sin(k_n h)} \cos k_n (z_0 - h) \sin(\omega t) e^{-k_n x_0} + \gamma_1 z_0 + \kappa_1 + \frac{a_n^2 k_n \omega}{2 \sin^2(k_n h)} \sin 2(\omega t) e^{-2k_n x_0}$$
$$- \frac{k_n a_n \omega}{\sin(k_n h)} \cos k_n (z_0 - h) \sin(\omega t) e^{-k_n x_0} (\gamma_1 z_0 + \kappa_1) t - \gamma_1 \frac{a_n}{\sin(k_n h)} \sin k_n (z_0 - h) \cos(\omega t) e^{-k_n x_0} \tag{15}$$

which indicates the Lagrangian velocity in upper fluid density including shear current.

(ii) *Lower Fluid Density:*

The horizontal velocity component is

$$u_2 = (\varphi_2)_x + U_2(z) = (\varphi_2)_x + \gamma_2 z + \kappa_2$$
$$= \frac{a_0 \omega}{\sinh(kh)} \cosh k (z - h) \sin(kx - \omega t) + \gamma_2 z + \kappa_2.$$

$$\therefore \frac{dx_2}{dt}\Big|_{(x=x_0, z=z_0)} = u_2 = \left(\frac{a_0 \omega}{\sinh(kh)} \cosh k (z - h) \sin(kx - \omega t) + \gamma_2 z + \kappa_2\right)\Big|_{(x=x_0, z=z_0)}$$

$$\therefore \int_{x_0}^{x} \frac{dx_2}{dt} dt = x_2 - x_{20} = \frac{a_0}{\sinh(kh)} \cosh k (z_0 - h) \cos(kx_0 - \omega t) + (\gamma_2 z_0 + \kappa_2) t \tag{14}$$

Similarly, the vertical velocity component is

$$\frac{dz_2}{dt} = w_2 = -\frac{a_0\omega}{\sinh(kh)}\sinh k(z-h)\cos(kx-\omega t)\Big|_{(x=x_0,z=z_0)}$$

$$\therefore \int_{z_0}^{z}\frac{dz_2}{dt}dt = z_2 - z_{20} = \frac{a_0}{\sinh(kh)}\sinh k(z_0-h)\sin(kx_0-\omega t) \tag{17}$$

The Lagrangian velocity at a point (x_0, z_0) is therefore,

$$U_{2L} = u_2(x_0, z_0, t) + (x_2 - x_{20})\left(\frac{\partial u_2}{\partial x}\right)_{(x_0, z_0)} + (z_2 - z_{20})\left(\frac{\partial u_2}{\partial z}\right)_{x_0, z_0,} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Applying various values in above equations and neglecting higher order terms, we get

$$\begin{aligned} U_{2L} &= \frac{a_0\omega}{\sinh(kh)}\cosh k(z_0-h)\sin(kx_0-\omega t)+\gamma_2 z_0+\kappa_2+\frac{ka_0^2\omega}{\sinh^2(kh)}\cosh^2 k(z_0-h)-\frac{ka_0^2\omega}{2\sinh^2(kh)} \\ &+\frac{ka_0^2\omega}{2\sinh^2(kh)}\cos 2(kx_0-\omega t)+\frac{ka_0\omega}{\sinh(kh)}\cosh k(z_0-h)\cos(kx_0-\omega t)(\gamma_2 z_0+\kappa_2)t \\ &+\gamma_2\frac{a_0}{\sinh(kh)}\sinh k(z_0-h)\sin(kx_0-\omega t) \end{aligned} \tag{18}$$

which indicates the Lagrangian velocity in lower fluid density including shear current.

Finally, we see that both equations (15) and (18) include shear current. But for upper fluid , it is exponential and trigonometric functions and for lower fluid , it is hyperbolic and trigonometric functions.

1.2. *The Velocity Field Decomposition Consists of a Potential Part but Not a Shear Current*

In that case, the shear current is absent, i.e., $\cup(z) = \gamma z + \kappa = 0.$

Therefore, the velocity components in terms of potential energy and stream function can be written as for upper fluid of density ρ_1,

$$\left.\begin{aligned} u_1 &= (\varphi_1)_x = (\psi_1)_z \\ w_1 &= (\varphi_1)_z = -(\psi_1)_x \end{aligned}\right\} \tag{19}$$

and for lower fluid of density ρ_2,

$$\left.\begin{aligned} u_2 &= (\varphi_2)_x = (\psi_2)_z \\ w_2 &= (\varphi_2)_z = -(\psi_2)_x \end{aligned}\right\} \tag{20}$$

For upper fluid, the horizontal velocity component is

$$u_1 = \frac{a_n\omega}{\sin(k_n h)}\cos k_n(z-h)\sin(\omega t)e^{-k_n x}$$

$$\therefore x_1 - x_{10} = -\frac{a_n}{\sin(k_n h)}\cos k_n(z_0-h)\cos(\omega t)e^{-k_n x_0} \tag{21}$$

Similarly, the vertical velocity component is

$$w_1 = \frac{a_n\omega}{\sin(k_n h)}\sin k_n(z-h)\sin(\omega t)e^{-k_n x}\Big|_{(x=x_0,z=z_0)}$$

$$\therefore z_1 - z_{10} = -\frac{a_n}{\sin(k_n h)} \sin k_n (z_0 - h) \cos(\omega t) e^{-k_n x_0} \tag{22}$$

The Lagrangian velocity at a point is found by Taylor expanding Eulerian velocity about (x_0, z_0). Therefore,

$$U_{1L} = u_1(x_0, z_0, t) + (x_1 - x_{10}) \left(\frac{\partial u_1}{\partial x}\right)_{(x_0, z_0)} + (z_1 - z_{10}) \left(\frac{\partial u_1}{\partial z}\right)_{x_0, z_0,} + ...$$

$$U_{1L} = \frac{a_n \omega}{\sin(k_n h)} \cos k_n (z_0 - h) \sin(\omega t) e^{-k_n x_0} + \frac{a_n^2 k_n \omega}{2\sin^2(k_n h)} \sin 2(\omega t) e^{-2k_n x_0} \tag{23}$$

which indicates the Lagrangian velocity in upper fluid density without shear current.

Similarly for lower fluid density, the horizontal velocity component is

$$u_2 = \left(\frac{a_0 \omega}{\sinh(kh)} \cosh k(z-h) \sin(kx - \omega t)\right)\Big|_{(x=x_0, z=z_0)}$$

$$\therefore x_2 - x_{20} = \frac{a_0}{\sinh(kh)} \cosh k(z_0 - h) \cos(kx_0 - \omega t) \tag{24}$$

and the vertical velocity component is

$$w_2 = -\frac{a_0 \omega}{\sinh(kh)} \sinh k(z-h) \cos(kx - \omega t)\Big|_{(x=x_0, z=z_0)}$$

$$\therefore z_2 - z_{20} = \frac{a_0}{\sinh(kh)} \sinh k(z_0 - h) \sin(kx_0 - \omega t) \tag{25}$$

Therefore, The Lagrangian velocity at a point (x_0, z_0) is found by

$$U_{2L} = \frac{a_0 \omega}{\sinh(kh)} \cosh k(z_0 - h) \sin(kx_0 - \omega t) + \frac{k a_0^2 \omega}{\sinh^2(kh)} \cosh^2 k(z_0 - h) - \frac{k a_0^2 \omega}{2\sinh^2(kh)} + \frac{k a_0^2 \omega}{2\sinh^2(kh)} \cos 2(kx_0 - \omega t) \tag{26}$$

which indicates the Lagrangian velocity in lower fluid density without shear current.

1. *Applications*

Case (i): With a Shear Current

A statement of upper fluid has been constructed in fluid dynamics. The averaged Lagrangian velocity can be written as

$$\cup_{1L} = \frac{1}{T} \int_0^T U_{1L} \, dt \quad \text{where } T = \frac{2\pi}{\omega}$$

$$= (\gamma_1 z_0 + \kappa_1) \left[1 + \frac{k_n a_n}{\sin(k_n h)} \cos k_n (z_0 - h) e^{-k_n x_0}\right] \tag{27}$$

With a shear current, a statement of lower fluid has also been constructed in fluid dynamics. The averaged Lagrangian velocity can be written as

International Journal of Innovation in Science and Mathematics
Volume 12, Issue 6, ISSN (Online): 2347–9051

$$\cup_{2L} = \frac{1}{T}\int_0^T U_{2L}\,dt$$

$$\cup_{2L} = (\gamma_2 z_0 + \kappa_2) + \frac{k a_0^2 \omega}{\sinh^2(kh)}\cosh^2 k(z_0 - h) - \frac{k a_0^2 \omega}{2\sinh^2(kh)} - \frac{k a_0}{\sinh(kh)}\cosh k(z_0 - h)(\gamma_2 z_0 + \kappa_2)\sin(k x_0) \quad (28)$$

Case (ii): Without a Shear Current

Without a shear current, a statement of upper fluid has been represented in fluid dynamics. Then the averaged Lagrangian velocity can be written as

$$\cup_{1L} = \frac{1}{T}\int_0^T U_{1L}\,dt = 0 \quad (29)$$

which shows that the aveaged Lagrangian velocity is zero without a shear current for upper fluid.

Also, for lower fluid, the averaged Lagrangian velocity is

$$\cup_{2L} = \frac{1}{T}\int_0^T U_{2L}\,dt = \frac{k a_0^2 \omega}{\sinh^2(kh)}\cosh^2 k(z_0 - h) - \frac{k a_0^2 \omega}{2\sinh^2(kh)} \quad (30)$$

If the vertical coordinate is zero, i.e., $z_0 = 0$ which shows that the flow is parallel to the horizontal axis, the averaged Lagrangian velocity is constant for lower fluid without a shear.

II. The Graphical Representations and Explanations

Mathematically, a graph shows the diagram for the solutions of problems and describes obviously the character of the given system analytically or numerically or also expresses for closed-form solution of the system. Also it needs construction of the basic acquaintance of a graph signified and we have shown two types of averaged Lagrangian velocities in different densities via this method. Therefore, some graphs of averaged Lagrangian velocities in different densities from the obtained solutions of Navier-Stokes equations generalized in section-3 through the variables separation method have been drawn within the figures 1-3 with the help of symbolic computation software, such as Mathematica.

For: $\gamma_1 = 1, \kappa_1 = 1, k_n = 1, h = -1, a_n = 1$

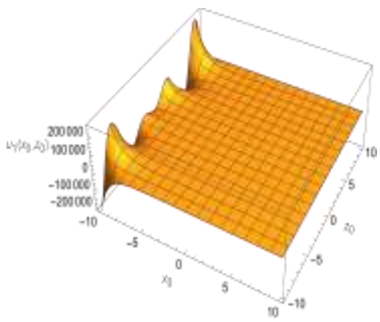

Fig. 1. Sketch of the Averaged Lagrangian veloicity of equation (27).

For: $\gamma_2 = 1, \kappa_2 = 1, k = 1, h = -1, a_n = 1, \omega = 1$

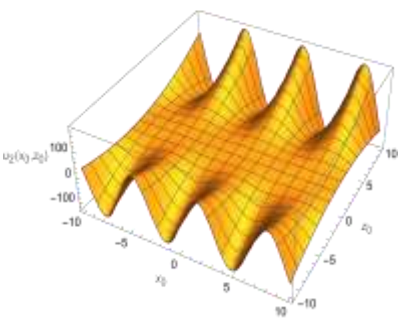

Fig. 2. Sketch of the Averaged Lagrangian veloicity of equation (28).

For: $k = 1; a_0 = 1; \omega = 1; h = -1$

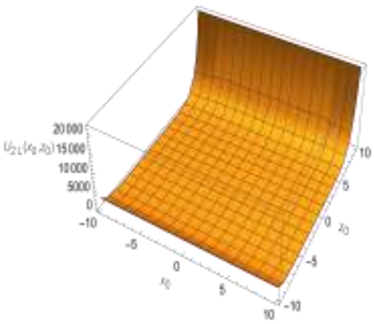

Fig. 3. Sketch of the Averaged Lagrangian veloicity of equation (30).

Figure-1 shows the averaged Lagrangian velocity for upper fluid and figure-2 shows the averaged Lagrangian velocity for lower fluid with a shear current. But Figure-3 shows the averaged Lagrangian velocity for lower fluid without a shear current and for upper fluid, it is zero.

III. CONCLUSIONS

The Navier-Stokes equations are governed by the surface waves in a layered fluid from which the Lagrangian velocities of upper and lower densities are derived consisting of a shear and not a shear respectively. Averaged Lagrangian velocities at two different densities have also been established which play a great significant role in fluid dynamics. Without a shear current the aveaged Lagrangian velocity is zero in upper fluid and in lower fluid it is constant. Also for lower fluid, if the vertical coordinate is zero, the flow is parallel to the horizontal axis.

REFERENCES

[1] Amitava Choudhuri, B., Talukdar, & Das, U.," Lagrangian Approach to Dispersionless KdV Hierarchy. Symmetry, Integrability and Geometry: Methods and Applications", SIGMA 3, 096, 2007, pp.11 pages.
[2] Nick Pizzo," Theory of deep-water surface gravity waves derived from a Lagrangian," J. Fluid Mech., vol. 896, 2020, A7.
[3] Bjornestad, M., Buckley, M., Kalisch, H., Streber, M., Horstmann, J., Froysa1, H.G., Ige1, O.E., Cysewski, M., & Carrasco-Alvarez, R. ,"Lagrangian Measurements of Orbital Velocities in the Surf Zone," Advancing Earth and Space Science. Geophysical Research Letters, 2021, 10.1029/2021GL095722.
[4] Constantin, A," A Lagrangian Approximation to the Water-Wave Problem," Applied Mathematics Letters, vol. 14, 2001, pp.789-795.
[5] Hongwu Zhao & Kamran Mohseni ,"A dynamic model for the Lagrangian-averaged Navier-Stokes-α equations," Physics of Fluids, vol. 17, 2005, 075106.
[6] Kamran Mohseni," The Lagrangian Averaged Navier-stokes Equations and optimization," Department of Aerospace Engineering University of Colorado, Boulder, 2009, FA9550-05-1-0334.
[7] Bhat, H.S., Fetecau, R.C., Marsden, J.E., Mohseni, K., & West, M.," Lagrangian Averaging for Compressible Fluids." Multiscale Model. Simul, Society for Industrial and Applied Mathematics, vol. 3, (4), 2005, pp. 818–837.
[8] Bhat, H.S., & Fetecau, R.C.,"Lagrangian Averaging for the 1D compressible Euler Equations," Discrete and Continuuous Dynamical Systems-Series , vol. 6(5), 2006, pp. 979–1000.
[9] Ambrosi, D.,"Hamiltonian Formulation for Surface Waves in a Layered Fluid," Waves Motion, vol. 31, 2000, pp.71-76.
[10] Marche, F.," Derivation of new two-dimensional viscous shallow water model with Varying Topography, Bottom Friction and Capillary Effects," European Journal of Mechanics-B/Fluids, vol. 26, 2007, pp.49-63.
[11] Longuet-Higgins, M.S.," Lagrangian Moments and Mass Transport in Stoke's Waves," Journal 1of Fluid Mechanics, vol. 179, 1987, pp. 547-555.
[12] Natalia Martinez Perez, & Ramirez, C.,"Lagrangian Formulation for Permionic and Super symmetric non-conservative systems. Journal of Physics Conference Series, vol. 654(1), 2015, 012005.
[13] Parvin, M.S., Alam Sarker M.S., & Sultana, M.S.,"| Mass Transport Velocity in a Two-dimensional Wave Tank,"App. Math. and Com. Intel., vol. 3(1), 2014, pp. 259-272.
[14] Parvin, M.S., Alam Sarker M.S., & Sultana, M.S.," Linearized Dispersion Relation of Navier-Stokes Equation in Shallow Water," International Journal of Scientific Engineering and Science, vol. 2 (1), 2018, pp. 83-91.
[15] Keyvan Sadeghi, "Lagrangian Potential Functions of Surface Forces and their Role in Fluid Mechanics,"Journal of the Brazilian Society of Mechanics Science and Engineering, vol. 45(8), 2023.
[16] Conor Curtin and Rossen Ivanov, "The Lagrangian Formulation for Wave Motion with a Shear Current and Surface Tension," J. Math. Fluid Mech., 2024, 25:87.
[17] Mst. Shahana Parvin, Mst. Shamima Sultana, "Lagrangian Formulation of Navier-Stokes Equations of Motions for the Surface Waves in Two-Layered Fluids," South Asian Journal of Biological Research (SAJBR), vol. 6(1), 2024, pp.1-14.
[18] Luc Deike Nick Pizzo and W. Kendall Melville,"Lagrangian transport by breaking surface waves," Journal of Fluid Mechanics, vol. 829, 2017, pp. 364-391.
[19] Didier Clamond, "On the Lagrangian description of steady surface gravity waves," Journal of Fluid Mechanics, vol. 589, 2007, pp. 433-454.
[20] Jan Erik H. Weber, Lagrangian studies of wave-induced flows in a viscous ocean," Deep Sea Research Part II: Topical Studies in Oceanography, vol. 160, 2019, pp.68-81.
[21] Umeyama, M.," Eulerian-Lagrangian analysis for particle velocities and trajectories in a pure wave motion using particle image velocimetry," Philosophical Transactions of the Royal Society A: Mathematical, Physical & Engineering Sciences, vol. 370(1964), 2012, pp. 1687-1702.
[22] Calvert, R., Whittaker, C., Raby, A., Taylor, P.H., Borthwick, A., & Van Den Bremer, T.," Laboratory study of the wave-induced mean flow and set-down in unidirectional surface gravity wave packets on finite water depth," Physical Review Fluids, vol. 4(11), 2019, 114801.
[23] Chen, Y.-Y., Hsu, H.-C., & Chen, G.-Y.,"Lagrangian experiment and solution for irrotational finite-amplitude progressive gravity waves at uniform depth," Fluid Dynamics Research, vol. 42(4), 2010, 045511.
[24] Lin, Y.-H., & Hwung, H.-H.," Infra-gravity wave generation by the shoaling wave groups over beaches,"China Ocean Engineering, vol. 26(1), 2012, pp. 1–18.

AUTHOR'S PROFILE

First Author
Mst. Shahana Parvin, Date of Birth: 25th June, 1979, Present Address: Professor, Department of Applied Mathematics, University of Rajshahi, Rajshahi-6205. Nationality: Bangladeshi, Medium of Education: English. Academic Qualificati--on: Year: 2002 (Held in 2004) Post Graduation: Master of Science (M. Sc.) in Applied Mathematics (Thesis) Institution : University of Rajshahi, Phone : 02588864155(Off), Mobile:+88-01710437663, Fax-0088-0721-750064, Rajshahi-6205, Bangladesh, Year: 2001 (Held in 2002) Graduation: B.Sc. Honors in Mathematics, Institution: University of Rajshahi, Rajshahi-6205. Field of Specialization: Current Main Research Field: Water Waves, Fluid Dynamics, Meteorology. Aw--ards/Scholarships (1) Scholarship in B.Sc. Program given by the University of Rajshahi, Bangladesh. (2) Scholarship in M.Sc. Program given by the University of Rajshahi, Bangladesh. (3) Scholarship for highest marks in Science Faculty given by Saida Khatun Merit

Scholarship, University of Rajshahi, Bangladesh. (4) Agrani Bank Gold Medal award for M.Sc. Program, University of Rajshahi, Bangladesh. Published Articles: Published 8(eight) research papers in National and International Journals. Computer Literacy: MATLAB, FORTRAN 77, 90, Programming with C, C++. Languages: Writing and speaking (fluently): English and Bengali, Arabic: Proficiency-Good.

Second Author

Md. Ashrafuzzaman Khan, Date of Birth: 1 April, 1980, Present Address: Professor, Department of Applied Mathema--tics, University of Rajshahi, Rajshahi-6205. Nationality: Bangladeshi. Medium of Education: English. Academic Quali--fication: Year: 2003 (Held in 2005) Post Graduation: Master of Science (M. Sc.) in Applied Mathematics (Thesis) Institution: University of Rajshahi, Rajshahi-6205, Bangladesh. Year: 2002 (Held in 2003) Graduation: B.Sc. H--onors in Mathematics Institution: University of Rajshahi, Rajshahi-6205, Phone: 02588864155 (Off) Mobile: +88-01717678291, Fax- 0088-0721-750064. Field of Specialization: Current Main Research Field: Water Waves, Fluid Dynamics, Meteoro--logy. Awards/Scholarships: (1) Scholarship in B.Sc. Program given by the University of Rajshahi, Bangladesh. (2) Scholarship in M.Sc. Program given by the University of Rajshahi, Bangladesh. Published Articles: Published 15(Fifteen) research papers in National and International Journals. Computer Literacy: MATLAB, Mathematica, FORTRAN, 77, 90, Programming with C, C++. Languages, Writing and speaking (fluently): English and Bengali. Arabic: Proficiency-Good. email id: akhanmath@yahoo.com