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The Nature of Imaginary Numbers from the Perspective of their “Dimensionality”

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Abstract **– The aim of this paper is to gain a better understanding of the nature of imaginary numbers, considering them as mathematical objects characterized by a lower dimensionality than that of real numbers. For this purpose, in addition to an article written a few years ago on the dimensional aspect of mathematical objects, reference will be made to the rules of arithmetic presented by Brahmagupta. In particular, it will be emphasized that even the nature of the arithmetic operation of multiplication has been misunderstood with regard to the roots of negative numbers. The negative numbers from which the square root is extracted are not to be seen as the result of a multiplication performed on the real straight line, i.e. without changing the nature of the multiplicand, but as the product between quantities of the same nature, a product that generates mathematical objects characterized by a nature different from that of the multiplicand. An alternative graphic approach to the usual one will finally better clarify the nature of imaginary numbers, highlighting their concreteness.**

Keywords **– Imaginary Numbers, N-Dimensional Mathematical Objects, Brahmagupta's Rules, Complex Plan.**

I. Introduction

Some years ago a paper was published on the different dimensional nature of mathematical objects [1]. The paper, while agreeing with Aristotle's view that lines are not composed of points [2], challenged his view that a line is generated by the movement of a point [3]. It should be pointed out that both of Aristotle's theses have been shared in modern times by various authors. However, in the cited paper [1] it is argued that for geometric objects (and in general also for other mathematical objects, such as sets) it is crucial to regard their dimensionality. The paper [1] states in particular that “A point has no dimension but it is not nothing”, that “There are no Points without N-Dimensional Geometric Objects (With $n > 0$)” and that “No one of Geometric Objects with at Least one Dimension is Generated by Points”. The generation process of geometric objects is therefore the opposite with respect to that usually proposed: a point (zero-dimensional) does not generate a line, but it is a position in a line or in a mathematical object with more dimensions: it exists if those objects exist. Moreover, mathematical objects with different dimensionality (for example: points and lines, lines and planes, but also sets and power sets) are not really comparable, “even if it is possible to put in biunivocal correspondence the points of a straight line, a plane or a volume with the points of a segment as small as you want”, because “they are only sets of points which we can identify into a segment, a straight line and so on, and not segments, straight lines and so on” [1].

In what follows, the theses proposed in the aforementioned paper concerning the dimensionality of mathematical objects will be taken into account in order to address the problem of the nature of a mathematical object that has long not been considered truly real: imaginary numbers.

II. Imaginary Numbers and Nature of Multiplication

Kronecker [4] said that God created natural numbers and that all the rest is a creation of man. Man's 'creations' could apparently include imaginary numbers (so called by Descartes [5]). The root of negative numbers had appeared, for the purpose of solving cubic equations by applying the Del Ferro-Fontana formula, in Cardano's "Ars Magna" [6] (thus violating the promise made to Fontana, better known as Tartaglia, not to publish the formula). The imaginary unit, successively named *i* by Euler (quoted by Merzbach and Boyer [7]), was introduced by Bombelli [8], who said it "numero sofistico" (sophistic number). However, Gauss (quoted by Bell [9]), to emphasize the full right of their existence, proposed another name for them: *lateral numbers*, due to their representation on the sides of the number line, in the Argand-Gauss (or complex) plane (actually, the complex plan had already been proposed by Wessel [10], but went almost unnoticed because it was presented in Danish). Gauss also suggested the name *direct numbers* and *inverse numbers* for positive and negative numbers.

However, while the use of imaginary (and complex) numbers has had increasingly wide applications thanks to the complex plane, the deep nature of these numbers has remained somewhat mysterious. In what follows, an attempt will be made to better define the nature of these numbers, showing that, by looking more closely at their dimensionality, they can reveal a nature that is anything but imaginary. For this purpose, it is opportune to go back at least to the arithmetic rules provided by Indian mathematicians.

Brahmagupta [11] was the first mathematician to write about the rules for doing arithmetic with zero and negative numbers. He named “fortunes” the positive numbers and “debts” the negative numbers. Therefore, following his approach, a debt subtracted from zero is a fortune and a fortune subtracted from zero is a debt.

As we know, in a multiplication, given two factors a and b, the first term (a) is the multiplicand, while the second (b) is the multiplier. In other words, the operation $a\ b$ tells us that a is added b times to 0 (or, in terms of sets, a elements must be inserted b times in the empty set). Multiplication is commutative: the product of $a\ b$ is equal to the product of $b\ a$, even if the operations are not exactly the same: in fact $a\ b$ tells us that a is added b times to 0, while $b\ a$ tells us that b is added a times to 0. So, the expression $(-a)\ b = -c$ tells us that the “debt” $-a$ is added b times to 0, giving the negative result $-c$, but we obtain the same result with $a\ (-b)$, which signifies that the “fortune” a is subtracted b times from 0. Finally, $-a\ (-b)$ indicates that the debt $-a$ is subtracted b times from 0, thus obtaining the positive value (fortune) c.

The result of these multiplications can be represented on the Cartesian axis of abscissas and this is a crucial point: multiplication usually does not change the nature of the objects, whose magnitude is measured by the term c, whatever its sign (fortune or debt, the unit of measurement is the same). Consequently the result c will always be represented on the straight line of real numbers.

III. Multiplication and the Dimensional Nature of Imaginary Numbers with Respect to Real Numbers

This does not happen for imaginary numbers: i represents the square root of -1, but we know that multiplying any number by itself, regardless of its sign, yields a positive value. Yet, approaching the problem, as done by Brahmagupta, from an economic point of view, and taking into account what sustained about the different nature of mathematical objects with different dimensionality [1], it is not difficult to imagine, to do a trivial example, a negative plot of land, in the sense that it still has to be worked both lengthwise and widthwise, compared to a plot of land that is already worked and is therefore active, in terms of work, for its owner. As mentioned above, multiplying a quantity a of objects by b, the nature of the result c is the same as a. This is not the case if the multiplicand and the multiplier are of the same or similar nature. For example, the product of a

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length by a length gives a result that can not be expressed in terms of length, but in terms of surface. They are all spatial quantities, but they are mathematical objects with different dimensionality that, according to Paolilli [1], are not comparable.

For purely explanatory purposes, we can therefore imagine real numbers as two-dimensional mathematical objects (like surfaces) and imaginary numbers as one-dimensional mathematical objects. Consequently, the straight line of real numbers will become a strip with a width of $\sqrt{-1}$ (that is *i*) from - ∞ to *0* or $\sqrt{1}$ from *0* to + ∞. Thus, real numbers will be represented by horizontal stripes and imaginary numbers by vertical segments (see Figure 1 and Figure 2; note that only the square root of a negative number must be indicated by a segment and therefore by a complex number).

It is no coincidence that in the complex plane multiplying a real number by *i* is equivalent to making a rotation of *90* degrees (counterclockwise), that is exactly half the rotation determined by the multiplication of two real numbers, the second of which (multiplier) is negative. In fact, to make an example, if real and imaginary numbers are two-dimensional and one-dimensional mathematical objects respectively, then $(-i)^4 = (-1)^2$.

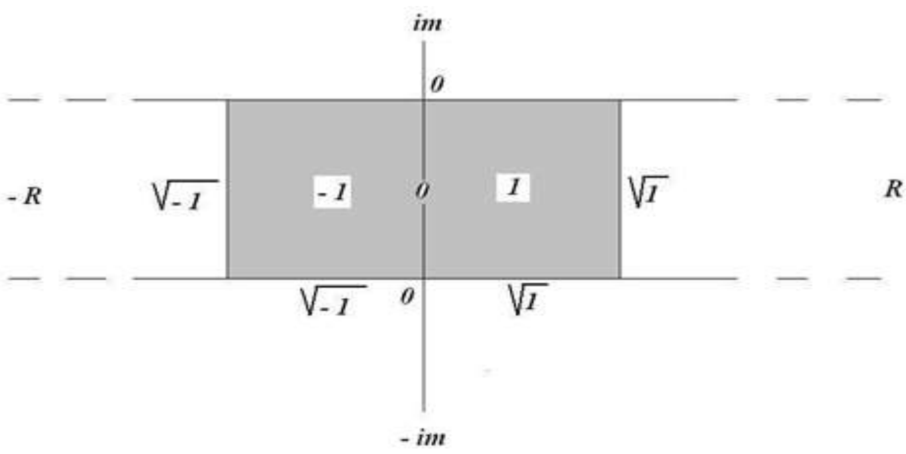

Fig. 1. Real numbers are measured on the strip: their absolute value increases as the distance from *0* (and thus the surface of the part of the strip that represents them) increases. Imaginary numbers are measured by segments above and below the strip. Their point *0* lies on the upper edge and the lower edge of the real numbers strip.

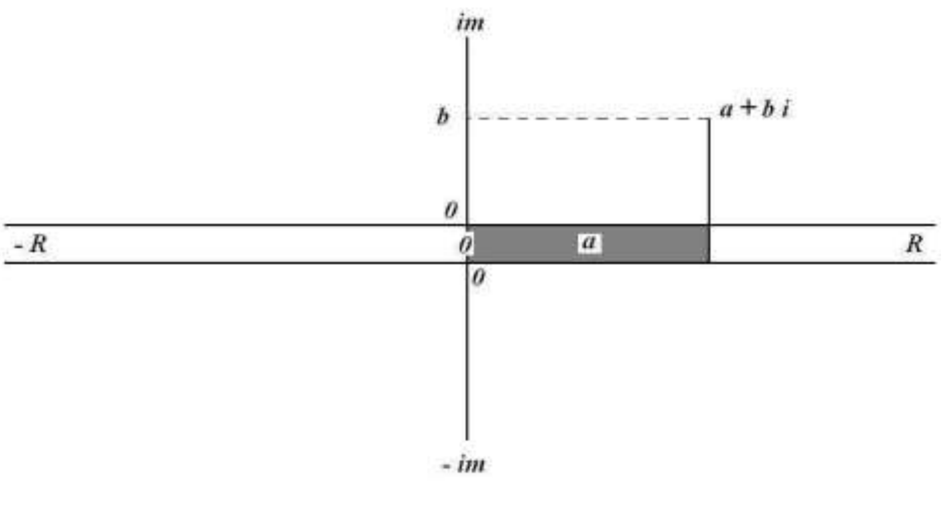

Fig. 2. An example of a representation of a complex number (*a* + *b i*) on a diagram like the one in Figure 1. As in the complex plane, the coefficient *b* is measured on the vertical axis.

However, it is more practical, and especially more in line with the current use of mathematics, to consider real numbers and imaginary numbers as one-dimensional and half-dimensional (that is elevated at *0.5*) objects respectively. In this way real numbers will be segments on the real number line, while imaginary numbers will be measured on the y-axis that cuts the real number line at point *0* (Figure 3).

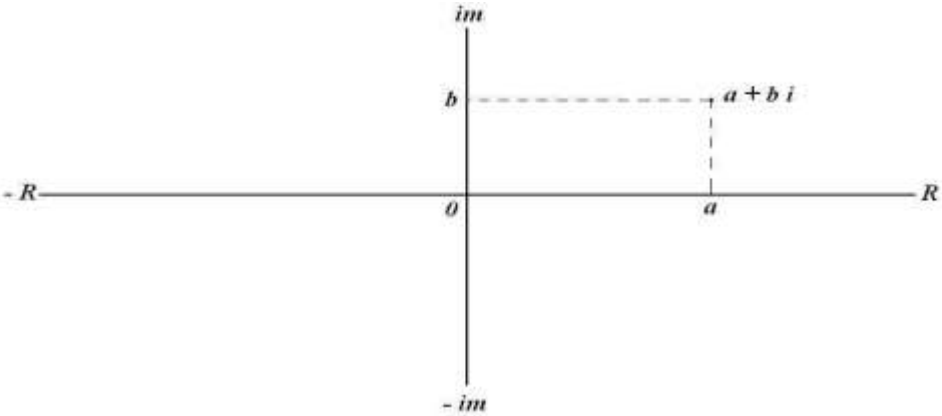

Fig. 3. In the complex plane the same complex number of Figure 2 is represented. The real number *a* is represented by the segment *0a*, while *b* is indicated by the (one-dimensional) segment *0b* which is, however, zero-dimensional with respect to the line of real numbers.

We have therefore returned to using the Argand-Gauss plane, although in this way the actual nature of the imaginary numbers may not be as evident as in Figure 2.

IV. Conclusion

The aim of this paper is to better understand the nature of imaginary numbers. To do this, it has been observed that they can be seen as mathematical objects characterized by having a lower dimensionality than real numbers. The basic arithmetic rules shown by ancient Indian mathematicians have also been used and analyzed. The conclusion is that imaginary numbers, whose symbol *i* could indicate a better-named "infranumeric unit", or infranumber, have the same level of reality as real numbers but have a lower dimensionality and therefore only their squares are truly comparable with real numbers.

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Antonio Luigi Paolilli, was born in Sulmona (Italy) on September 15, 1958. Expert in Economics at the Salento University (Italy), PhD in Historical-Geographical Studies and International Relations. He is the author of papers, published in various journals, with mathematical models for the study of cooperation, economic history and economic geography. For the IJISM he is the author of several papers on Mathematics and Relativity.