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The Behavior of the Quadratic Function $y = ax^2 + bx + c$, with $a, b, c \in R$ and $a \neq 0$ According to the Variation of the Coefficient B

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Abstract – **This paper seeks to demonstrate a particular property of the quadratic function. It describes the influence of each of the expression coefficients in $y = ax^2 + bx + c$, with $a, b, c \in$ R and $a \neq 0$ regarding the graph of the parable and, in particular, analysing the influence of the coefficient b, detailing the behavior of the vertex of any quadratic function according to the variation of b. We conclude that the vertex variation draws another quadratic function in which the resulting parable has concavity in the opposite direction and symmetry to the y axes.**

Keywords – **Demonstration, Parable, Quadratic Function, Coefficients, Parable Vertex.**

I. Introduction

Function [1], in mathematics, is a fundamental tool created for the study of quantitative laws. A particular case of this broad concept is the quadratic function whose general analytical expression is $y = ax^2 + bx + c$, with $a, b, c \in$ R and $a \neq 0$ and whose graph in the Cartesian system is a parable. Studying this kind of function is a compulsory component in mathematics, especially in High School as well as in Elementary School [2]. But in Brazil, for example, due to the lack of investment in a better pedagogical method to teach mathematics, focusing mostly on memorizing rather and understanding, gives Brazil a very low position in comparison to other developing nations, like Mexico and Chile, in major worldwide mathematics rankings, like OECD PISA [3]. Providing a deeper understanding of mathematical concepts through analysis and application since early stages of education if very important for a country's educational development. And since quadratic functions and their behavior always give a certain fascination in teachers and students because of its fundamental practical utility in various branches of science, as well as for its graphical elegance, it is important to provide a broad understanding of it in High School. In textbooks and in classrooms we have various references to the behavior of quadratic functions according to the variation of the coefficient *a* & *c*. However, little or even no emphasis is given to the importance of the coefficient *b*, as well as what are the graphic consequences of its variation. This creates a gap for the proper understanding of quadratic functions and its behavior. Therefore, by first analyzing the behavior of the parable of a quadratic function according to the variation of the coefficient *b*, we observe that it delineates, through the vertex of the original parable, another inverse quadratic function. This new quadratic function, which we will call Vertex-Function, follows the usual behavioral patterns when *a* & *c* vary, but since *b* coefficient is null, this function always has its vertex fixed at the *y* axis, (0, y).

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II. THE VARIATION OF THE COEFFICIENT B

By studying this variation, it is possible to see that the parable described by the vertex "movement", when the coefficient *b* is changed, it has equal concavity to that of the displaced parable, but in the opposite direction, i.e. one with downward concavity, while the other with upward concavity, symmetrical to the Y axis with its vertex always at the point (0, y). Thus, the following statement can be made: if a parable defined by $y = ax^2 + bx + c$, with $a \neq 0$ e $a, b, c \in R$ has its coefficients *a* and *c* fixed with certain number each, but change in the coefficient *b*, then its vertex will be translated describing a new parable.

Returning to the starting point to bring the concepts necessary for the demonstration to follow, the following summary is made: the parable has its configuration determined by the coefficients of the function, and the coefficient *a* is related to its concavity; the coefficient *c*, also called linear, displaces the parable with regard to the Y axis. Whereas the coefficient *b* exerts its influence on the displacement of the parable vertex.

To Prove this we go through two Distinct Stages or Moments:

In the first moment, the graph is drawn, showing the displaced elements of the vertex according to the variation of the coefficient *b*. By analyzing the resulting graph we try to identify the resulting curve, made by the vertex, and deduce the mathematical expression that defines this new function. It is verified, in the graphical display of the function, that the coordinate points x = 0 and y = c remain constant. Therefore, this is a common point of all graphs for this family of parables, where *a* and *c* are constant, but *b* varies according to the set of Real numbers. With this conclusion the mathematical expression of the new function is defined. It constitute a new quadratic function with its vertex at the point (0, c), which makes the function symmetrical to the Y axis.

In the second moment, the demonstration proceeds using the properties of algebra in two other moments. In the first one it is proved, in two ways, that every point, determined by the translating vertex of a quadratic function where *b* varies, is a point in this new resulting quadratic function. In the second moment, the opposite is proved, that is, that any point in this resulting parable is one of the points by which the vertex of the first function makes its way.

These demonstrations are labeled as inductive or ascending because it went from a particular case to a generic application to all quadratic functions. As the aforementioned author also states, "in mathematics, the truth criterion is non-contraction, and induction, that gives rise to new truth criterion, discovers new mathematics relations, grounded on the combination of previous concepts" [4].

III. ANALYSIS AND DEMONSTRATION OF THE BEHAVIOR OF QUADRATIC FUNCTIONS ACCORDING TO THE VARIATION OF THE COEFFICIENT B

The question that guides this paper is the following: What is the function drawn by the vertex of any quadratic function according to the variation of the coefficient *b*? Figure 1 helps us visually understand the behavior of this function.

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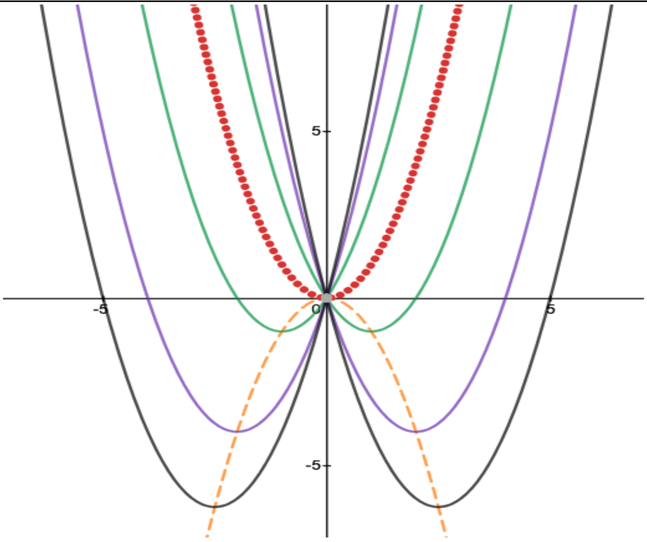

Fig. 1. The red dotted quadratic function $y = x^2 + 0x + 0$, by changing its coefficient *b*, (examples: 2, 4, 5, -2, -4, -5) forms the inverted parable, with orange dashes.
Source: https://www.desmos.com/calculator.

Understanding that the resulting function drawn by the vertex of a quadratic function according to the variable *b* is a new parable, we take the formulas for obtaining the vertex values [5], $X(v) = - b/2a$; $Y(v) = - \Delta/4a$

By isolating the values of the coefficient *b* in X (v) (x of the vertex), we have: $b = -2aX$

Doing the same with Y (v), (y of the vertex):

$$Y(v) = - \frac{b^2 - 4ac}{4a}$$

$$Y = - \frac{b^2}{4a} + c$$

$$Y - c = - \frac{b^2}{4a}$$

$$4a\,(Y - c) = - b^2$$

We have, $b = \sqrt{-4aY + 4ac}$

By equating the terms, we can isolate the values of Y as a function of X to generate a vertex function according to the variation of the coefficient *b*.

$$-2aX = \sqrt{-4aY + 4ac}$$

$$(-2aX)^2 = -4aY + 4ac$$

$$\frac{4a^2X^2}{4a} = \frac{-4aY + 4ac}{4a}$$

$$aX^2 = -Y + c$$

Finally, we arrive at the equation that describes the variation of the vertex of any quadratic function according to the variation of its coefficient *b*. We will call this function the Vertex Function: $y = -_a X^2 + c$

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Another way to achieve the same result can be done as follows: Let the family of parables defined by $y = a_{\circ}x^2 + b_{\circ}x + c_{\circ}$ whose vertices have coordinates, $(Xv\,,Yv)$ which are $(\frac{-b_{\circ}}{2a_{\circ}}\,,\frac{-b^2{}_{\circ}+4a_{\circ}c_{\circ}}{4a_{\circ}})$ with $a_{\circ}, b_{\circ}, c_{\circ}\ \in R, a_{\circ} \neq 0$. Graphically these vertices form other parables with vertices in $(0, c_{\circ})$ and defined by $y = \alpha x^2 + \beta x + c_{\circ}$ being $\alpha = -a_{\circ}\,, \beta = 0$. That is, $y = -a_{\circ}x^2 + c_{\circ}$, as will be verified below:

If we take any three functions of the given family considered, such that:

$y = a_{\circ}x^2 + b_1x + c_{\circ}$, with vertices $(\frac{-b_1}{2a_{\circ}},\frac{-b_1{}^2+4a_{\circ}c_{\circ}}{4a_{\circ}})$

$y = a_{\circ}x^2 + b_2x + c_{\circ}$, with vertices $(\frac{-b_2}{2a_{\circ}},\frac{-b_2{}^2+4a_{\circ}c_{\circ}}{4a_{\circ}})$

$y = a_{\circ}x^2 + b_3x + c_{\circ}$, with vertices $(\frac{-b_3}{2a_{\circ}},\frac{-b_3{}^2+4a_{\circ}c_{\circ}}{4a_{\circ}})$

Begin $y = \alpha x^2 + \beta x + c_{\circ}$ the function whose graph consists of the vertices of the given family parables, let's prove that $\alpha = -a_{\circ}\,, \beta = 0$. Substituting in x and y the coordinates of the vertices in the three family functions we have:

I. $\alpha\left(\frac{-b_1}{2a_{\circ}}\right)^2 + \beta\left(\frac{-b_1}{2a_{\circ}}\right) + c_{\circ} = \frac{-b_1{}^2+4a_{\circ}c_{\circ}}{4a_{\circ}}$

II. $\alpha\left(\frac{-b^2}{2a_{\circ}}\right)^2 + \beta\left(\frac{-b^2}{2a_{\circ}}\right) + c_{\circ} = \frac{-b^{22}+4a_{\circ}c_{\circ}}{4a_{\circ}}$

III. $\alpha\left(\frac{-b_3}{2a_{\circ}}\right)^2 + \beta\left(\frac{-b_3}{2a_{\circ}}\right) + c_{\circ} = \frac{-b_3{}^2+4a_{\circ}c_{\circ}}{4a_{\circ}}$

Eliminating the denominators we have:

$$\alpha b_1{}^2 - 2\beta b_1 a_{\circ} + 4a_{\circ}{}^2c_{\circ} = -a_{\circ}b_1{}^2 + 4a_{\circ}{}^2c_{\circ}$$

$$\alpha b_2{}^2 - 2\beta b_2 a_{\circ} + 4a_{\circ}{}^2c_{\circ} = -a_{\circ}b_2{}^2 + 4a_{\circ}{}^2c_{\circ}$$

$$\alpha b_3{}^2 - 2\beta b_3 a_{\circ} + 4a_{\circ}{}^2c_{\circ} = -a_{\circ}b_3{}^2 + 4a_{\circ}{}^2c_{\circ}$$

Multiplying II by (-1) and having I + II we have:

$$\alpha b_1{}^2 - \alpha b_2{}^2 - 2\beta a_{\circ}(b_1 - b_2) = -a_{\circ}b_1{}^2 + a_{\circ}b_2{}^2$$

$$\alpha\left(b^{1^2} - b^{2^2}\right) - 2\beta a_{\circ}(b^1 - b^2) = -a_{\circ}(b_1{}^2 - b_2{}^2)$$

$$\alpha(b_1 - b_2)(b_1 + b_2) - 2\beta a_{\circ}(b_1 - b_2) = -a_{\circ}(b_1 - b_2)(b_1 + b_2)$$

We then divide the expression by $(b_1 - b_2)$:

$$\alpha(b^1 + b^2) - 2\beta a_{\circ} = -a_{\circ}(b_1 + b_2)$$

$$(\alpha + a_{\circ})(b_1 + b_2) = 2\beta a_{\circ}$$

$$\frac{(\alpha + a_{\circ})(b_1 + b_2)}{2a_{\circ}} = \beta$$

Replacing it in I

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$$\alpha {b_1}^2 - \frac{2(\alpha + a_\circ)(b_1 + b_2)b_1 a_\circ}{2a_\circ} = -a_\circ {b_1}^2$$

$$\alpha {b_1}^2 - (\alpha b_1 + \alpha b_2 + a_\circ b_1 + a_\circ b_2)b_1 = -a_\circ {b_1}^2$$

$$\alpha {b_1}^2 - \alpha {b_1}^2 - \alpha b_1 b_2 - a_\circ {b_1}^2 - a_\circ b_1 b_2 = -a_\circ {b_1}^2$$

$$-\alpha b_1 b_2 = a_\circ b_1 b_2$$

$$\alpha = -a_\circ$$

Applying it in III

$$-\alpha a {b_3}^2 - 2\beta b_3 a_\circ = -a {b_3}^2$$

$$-2\beta b_3 a_\circ = 0$$

As $a_\circ \neq 0$

$$-\beta b_3 = 0$$

As b_3 is not necessarily null, therefore $\beta = 0$. For the following function, $y = \alpha x^2 + \beta x + c_\circ$, therefore, we also arrive at the Vertex Function: $\boldsymbol{y = -a_\circ x^2 + c_\circ}$

IV. The Influence of the Coefficients A and B in the Vertex Function

The parable has its concavity changed if *a* is positive or negative. This coefficient also determines whether the concavity has a bigger or smaller overture. [6] If $a > 0$ then the parable will have its concavity upwards; if $a < 0$ then it will be downwards, as shown in figure 2. Another aspect regarding *a* is that, if $0 < a < 1$, then the parable will have a larger overture, tending to become a straight line as *a* tends to 0. If $a > 1$, the opposite happens, once the overture of the parable becomes smaller as it tends to infinity. In both cases, the graph of the function will look more and more like a semi straight line, as seen in figure 3.

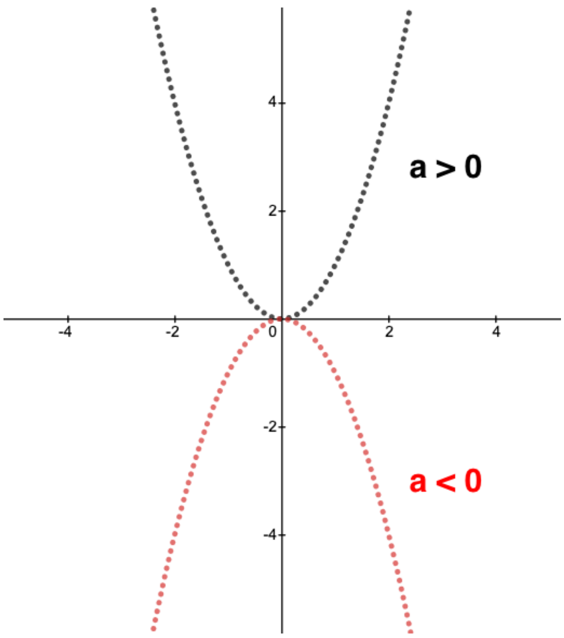

Fig. 2. Behavior of the parable for *a > 0* and *a < 0*.
Source: https://www.desmos.com/calculator.

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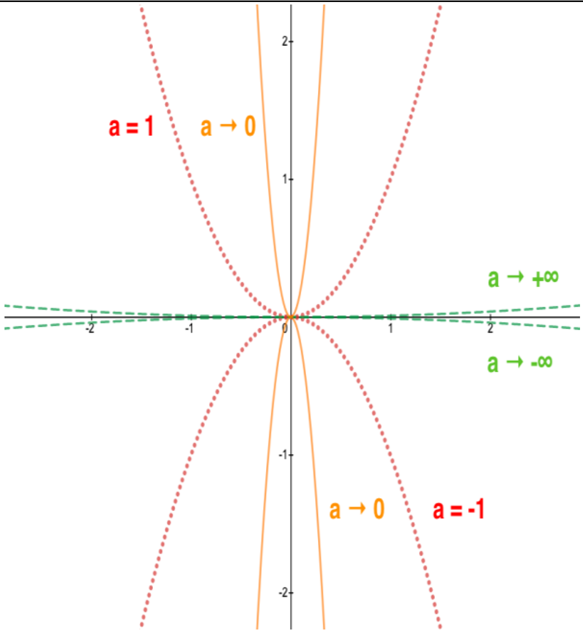

Fig. 3. Behavior of the parable when *a* tends to zero or when *a* tends to infinity.
Source: https://www.desmos.com/calculator.

The same way, the variation of the coefficient *c* moves the parable up or down along the Y axis, not changing its shape, (Figure 4). This mathematical attribute, as well as the behavior of the quadratic function regarding the variation of *a,* is widely taught in High Schools worldwide, being available in most didactic math books [7].

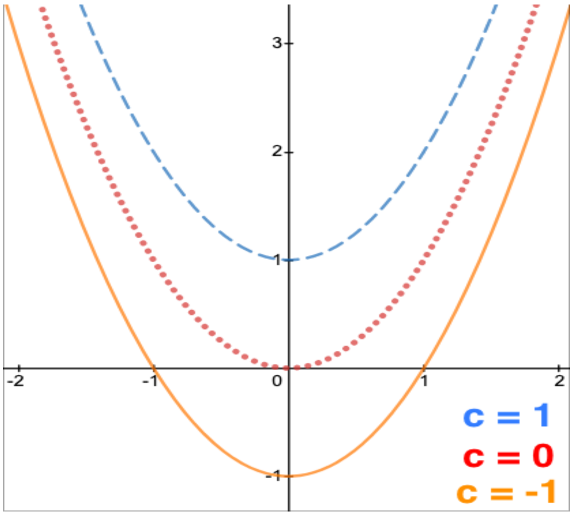

Fig. 4. Behavior of the parable according to the variation of *c*.
Source: https://www.desmos.com/calculator.

When we apply it to the Vertex Function, ($y = -ax^2 + c$), since its coefficient *b* is null, we observe that it is symmetrical at the Y axis, always positioned at (0, c), going up or down according to *c,* obeying the laws of behavior for the variation of its coefficients [8].

V. Conclusion

Regarding the behavior of any quadratic function by the variation of its coefficient *b*, we are able to conclude the existence of a relevant mathematical attribute related to the study and analysis of second degree polynomial functions. This variation draws, with the vertex of the parable, a new parable according to $y = -ax^2 + c$. The

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analysis of this new quadratic function related to the vertex gives a new perspective and a better comprehension of the behavior of quadratic function beyond the usual variation of the coefficients *a* or *c,* which are already taught widely and is well documented.

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