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On the Existence and Application of Hausdorff Space

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***Abstract* – Hausdorff space is a topological space a separation property in that, any two distinct points can be separated by a disjoint open set - that is, whenever x and y are distinct points of a set X, then there exist disjoint open sets U_x and V_y such that U_x contains x and V_y contains y and $U \cap V = \emptyset$. This study aims at making Hausdorff Space real and stating some of its applications. The study employed the bonding of inert gases and fully describes the various types of bonding in the context of topology. The study compares two inert gases at a time, checking all possible ways to see if they can be bonded. That is, the two inert gases picked are two distinct points in the set X such that their intersection gives a null set. Because of their nonreactive nature and fully filled valence shells, the two inert gases cannot be bonded. Also, one application of Hausdorff Space was seen in controlling the COVID-19 pandemic, that is, social distancing. This is possible if we let two people say, x and y be in a room X together and they are not allowed to share anything in common and are to maintain a particular distance, say 1.5 metres to 2 metres, throughout the room. Then x and y are two distinct points of sets in the set X. This is social distancing and Hausdorff Space.**

***Keywords* – Hausdorff Space, Topological Space, Distinct point, Bonding, Inert Gas.**

I. INTRODUCTION

Analytical thinking, logical reasoning, problem-solving skills, and precise communication are all required in mathematics as a field of study. The mathematical language is used in every field of science because of its power and versatility. Topology is a fascinating branch of mathematics that not only introduces new concepts and theorems but also puts old ones, such as continuous functions, into context. To say only this, however, is to minimize the importance of topology [19].

Topology is one of the most active disciplines in mathematics because of its applications in a wide range of fields, including differentiable equations, dynamical systems, knot theory, and Riemann surfaces in complex analysis. It's also used in physics to describe the universe's space-time structure and in string theory. However, because of the abstract concepts involved, students perceive it to be difficult, and those who do prefer or come across it believe that all they need to do is memorize to pass their exam [20]. Furthermore, because topology is an abstract concept, discussing the unknowns makes mathematics appear to be a difficult subject for everyone, [25]. Other studies were successful in explaining related concepts like metric space, hamming distance, and Levenshtein distance using real-world examples. The applicability of metric space was also demonstrated using DNA hybridity [29]. Amoeba Proteus, a single-celled organism, was also successfully used to demonstrate the use of Homeomorphism [28]. However, some topological concepts, such as Hausdorff space, continue to be a source of anxiety for students. Such concepts appear to be extremely difficult to grasp in research works. Some of these issues could be resolved if some of these abstract concepts came to life and their applications were demonstrated.

As a result, the goal of this paper is to prove the existence of a concept in general topology known as Hausdo-

-rff Space and to discuss some of its applications.

II. Preliminaries

2.1. *Topological Space*

A topological space is a three-dimensional space that is a generalization of the concept of an object. It is made up of an abstract set of points and a specified collection of subsets, known as open sets, that satisfy three axioms:

(i) The set itself and the empty set are open sets;

(ii) The intersection of a finite number of open sets is open; and

(iii) The union of any collection of open sets is an open set.

2.1.1. *Definition*

Let (X, τ) be any topological space, then the members of τ are said to be open sets [19].

2.2. *Hausdorff Space*

A Hausdorff space is a topological space with a separation property in that, any two distinct points can be separated by disjoint open sets. That is, whenever x and y are distinct points of a set X then there exist a disjoint open sets U_x and V_y such that U_x contains x and V_y contains y.

2.2.1. *Theorem*

Every two distinct points in a space are Hausdorff if their neighbourhoods are disjoint. [8].

2.2.2. *Definition*

A topological Space (X, τ) is said to be a Hausdorff Space if given any two pair of distinct points x, y in X, there exist an open set U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$ [16].

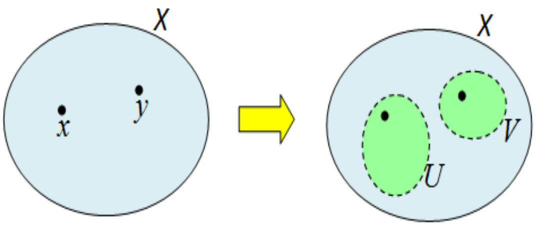

Fig. 1. Two pair of distinct points (x, y) in X.

2.2.3. *Example*

Let (X, d) be any metric space and τ be the topology induced on X by d. Then (X, d) is a Hausdorff Space. This is because, if we let x and y be any point on X, with $x \neq y$, then $(x, y) > 0$. Putting $\xi = d(x, y)$, we consider the open balls $B_{\xi/2}(x)$ and $B_{\xi/2}(y)$. This is an open set in (X, τ) with $x \in B_{\xi/2}(x)$ and $y \in B_{\xi/2}(y)$. But for τ to be a Hausdorff Space, $B_{\xi/2}(x) \cap B_{\xi/2}(y) = \emptyset$.

Suppose $Z \in B_{\xi/2}(x) \cap B_{\xi/2}(y)$, then $d(z,x) < \frac{\xi}{2}$ and $d(z,y) < \frac{\xi}{2}$.

Hence; d(x,y) $\leq d(x,z) + d(z,y)$

$< \frac{\xi}{2} + \frac{\xi}{2} = \xi$

$d(x,y) < \xi$

This is not true since we set $\xi = d(x,y)$. Therefore, there exists no x in $B_{\xi/2}(x) \cap B_{\xi/2}(y)$; that is $B_{\xi/2}(x) \cap B_{\xi/2}(y) = \emptyset$.

2.3. *Metric Space*

2.3.1. *Definition*

Let X be a non-empty set and d be a real-valued function defined on $X \times X$ such that for $a, b \in X$;

(i) $d(a,b) \geq 0$ *and* $d(a,b) = 0$ *if and only if* $a = b$ *(positive definite);*

(ii) $d(a,b) = d(b,a)$ *(symmetric);*

(iii) $d(a,c) \leq d(a,b) + d(b,c)$ *(Triangular inequality).*

Point *(iii)* is a triangular inequality since it represents the geometrical reality that one of the triangle's sides is less than or equal to the sum of the other two sides' lengths. Because all of a, b and c are in X, d is termed a metric on X, (X,d) is called a metric space, and $d(a,b)$ is called the distance between a and b. We normally say X is the metric space but if we wish to be more specific, we write (X,d) as the metric space [29].

2.3.2. *Examples*

(i) Both R and C are metric spaces when endowed with the distance function $d(a,b) = |a - b|$. We will often refer to “the metric space R”, without explicitly mentioning the metric.

(ii) Let $F: R \to R$ be any strictly monotonic function (say increasing), then R endowed with the distance function $d(a,b) = |f(a) - f(b)|$ is a metric space. Symmetry and positivity are immediate. It only remains for us to verify the triangular inequality. For $a, b, c \in R$, $d(a,c) = |f(a) - f(c)| \leq |f(a) - f(b)| + |f(b) - f(c)|$.

2.3.3. *Theorem*

Every metric space is a Hausdorff Space.

2.4. *Chemical Element*

2.4.1. *Definition*

An element is a material made up of the same type of atoms that can't be broken down into simpler substances using any known chemical method. It's possible to think of it as matter's fundamental chemical building block. It is a chemical compound made up of only one sort of atom. Examples of Chemical Elements are; Hydrogen (H), Lithium (Li), Neon (Ne), Magnesium (Mg), etc. The number of protons in an element's nucleus is referred to as its atomic number (represented by the symbol Z), and all atoms with the same atomic number are atoms of the s-

-ame element.

2.4.2. *Definition*

The smallest portion of an element that can participate in a chemical reaction is an atom. [3].

2.4.3. *Definition*

A chemical reaction occurs when one substance (the reactant) is transformed into another material (the product). Chemical elements and compounds are both substances. The constituent atoms of the reactants are rearranged in a chemical reaction, resulting in distinct compounds as products. [22].

2.4.4. *Definition*

The number of protons found in the nucleus of a chemical element's atom is its atomic number (proton number). [23].

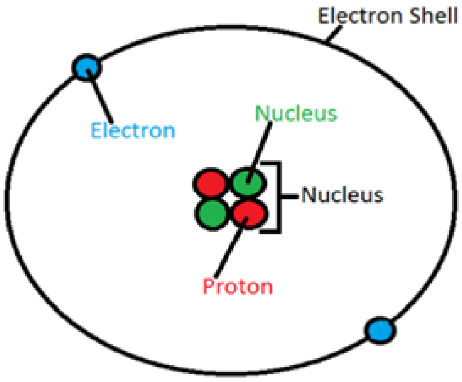

Fig. 2. Chemical element showing nucleus of the atom.

III. Main Results

3.1. *Ionic Bonding Between Inert Gasses*

Between a positively charged ion and a negatively charged ion, there is an electrostatic force of attraction in an ionic bond. A cation and an anion make up ionic bonding. When an atom, usually a metal, loses an electron or electrons and becomes a positive ion (cation), another atom, usually a non-metal, is able to gain the electron (s) and become a negative ion, hence a bond is created (anion).

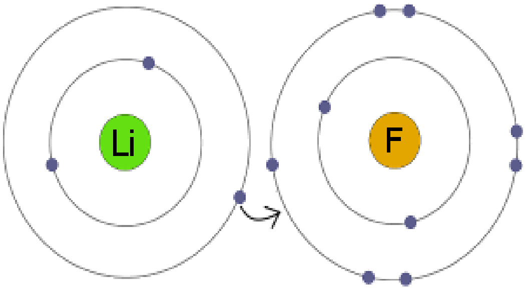

Fig. 3. Ionic Bonding between Lithium (Li) and Fluorine (F).

In the case of inert gasses, they are all chemically unreactive due to the stable complete shall of electrons (Doublet or Octet State). Thus, in the table below, they all have outer shells fully filled.

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3.2. *Covalent Bonding between Inert Gasses*

In covalent bonding, there is a sharing of an electron pairs between the atoms involved to give rise to a stable electron configuration of the bonded species. Covalent bonds are formed mostly between non-metals. To achieve the noble gas configuration, the atoms involved in the bond share valence electrons. That is, one electron is denoted by each atom which is held in common by both atoms.

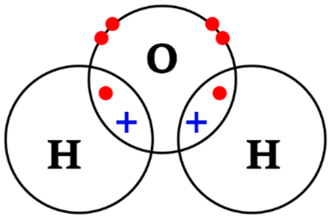

Fig. 4. Covalent bonding between Hydrogen and Oxygen.

In the case of two inert gasses say Argon and Krypton, they all attain the octet state and hence there is nothing left for them to share to attain that state. Looking at the structure of the two atoms below, we can clearly see that, they all have a fully filled valence shall which will automatically prevent covalent bonding between them. Krypton with a fully filled and Argon with a fully filled valence shall have no electron in their outer shell that makes them unstable hence they can't have a shared pair of electrons and for that matter, they cannot form covalent bonds.

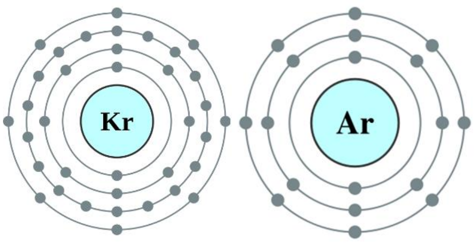

Fig. 5. Representation of the Electronic Configuration of Kr and Ar.

3.3. *Hausdorff Space and the Inert Gasses*

A Hausdorff space is a topological space with a separation property in that, any two distinct points can be separated by disjoint open sets. That is, whenever x and y are distinct points of a set X, then there exist disjoint open sets U_x and V_y such that U_x contains x and V_y contains y.

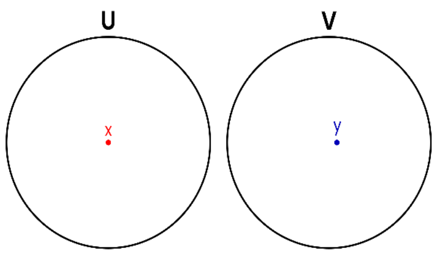

Fig. 6. Disjoint Open Sets U_x and V_y.

A topological Space (X, τ) is said to be a Hausdorff Space if given any two pair of distinct points x, y in X, there exists an open set U and V such that $x \in U, y \in V$, and $U \cap V = \emptyset$.

3.3.1. *Theorem*

Every metric space is Hausdorff, especially $\mathfrak{R}^n$ is Hausdorff (for $n \geq 1$) [29].

Proof:

Let (X, d) be a metric space and let $x, y \in X$ with $x \neq y$. Let $r = d(x, y)$. Let $U = B(x: r/2)$ and $V = B(y: r/2)$. Then $x \in U$, $y \in V$. We claim $U \cap V = \emptyset$. If not there exists $z \in U \cap V$. But then $d(x, z) < r/2$ and $d(y, z) < r/2$ so we get; $r = d(x, y) \leq d(x, z) + d(z, y) < r/2 + r/2$ i.e. $r < r$, a contradiction. Hence $U \cap V = \emptyset$ and X is Hausdorff.

3.3.2. *Example*

Let (X, τ) be a topological space, and let x, y be some distinct points that are in X, then there exists some open sets U and V such that $x \in U$ and $y \in V$. Let the open set U be one of the inert gasses say Xenon and also let V be another inert gas say Radon.

Let the electrons of Xenon be x together with its protons and neutrons in the nucleus of the atom and let the electrons of Radon be y together with its protons and neutrons in the nucleus.

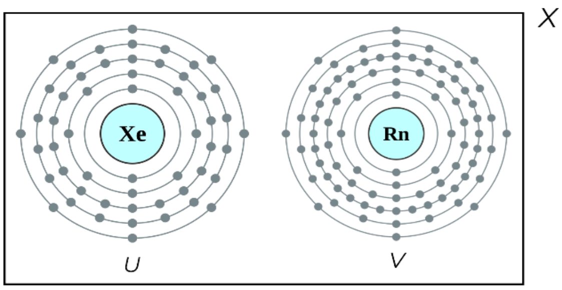

Fig. 7. Representation of the Set Xenon (Xe) and Radon (Rn) in *X*.

From the definition of Hausdorff Space, if we have any two pair of distinct points x, y in X, there exist an open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In the diagram above, the sets U (Xenon) and V (Radon) are two distinct sets in X, such that $U \cap V = \emptyset$. That is; x, y in X, $\exists$ x (electrons of Xenon) $\in U$ and y (electrons of Radon) $\in V$ such that $U \cap V = \emptyset$. This is so since when forming a compound or bonding Xenon and Radon together is not possible.

In fig. 7, the sets Xenon and Radon are two open sets with two distinct pointsx, y in X such that $x \in U$ and $y \in V$. If these two sets are taken through covalent bonding, they will have no electrons to share which implies that the intersection between them is empty. That is; $U \cap V = \in \emptyset$. Using alternative way; Let x be the spins of Helium and y be the spins of Neon.

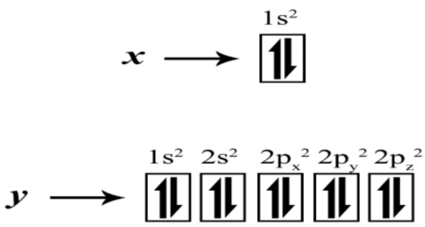

Fig. 8. The spins of Helium and Neon.

From the above representations, it can be seen that the electrons fully filled the orbitals hence both elements will neither give nor receive any other electrons. Now if there is an open set U and V such that $x \in U, y \in V$, then the intersection of U and V is empty ($\emptyset$). That is, $U \cap V = \emptyset$.

3.4. *Application of Hausdorff Space*

The COVID-19 pandemic brought many things, some of which were the result of the application of some mathematical concepts. Topology, as a branch of mathematics, is no exception. Some protocols, such as handwashing with soap under running water, wearing face masks, and social distancing, are to be followed in order to limit the spread of this pandemic.

In one of these protocols, social distancing, Hausdorff space is visible. Assume two people are supposed to be standing in the middle of a room. If a circle is drawn around each of them in such a way that the circles do not overlap, that is, the circles around each individual do not intersect each other, the room is transformed into a Hausdorff space. The room is said to be a Hausdorff space if each of the circles has a radius of 0.75 meters to 1 meter and a minimum space between them, and if each of them moves, it moves with the circle, preserving the radius and space. As a result, the distance between them will be between 1.5 and 2 meters at any point in time and in any position in the room. When it comes to controlling the spread of COVID-19, the Ghana Health Service recommends a social distance of 1.5 to 2 meters. This is Hausdorff space, a concept widely used in topology, a branch of mathematics used to maintain social distance.

Mathematically;

Let the room be represented by the set X, also let the first person in the room be x and the second person be y. Similarly, let the first and second circles be U and V respectively, then;

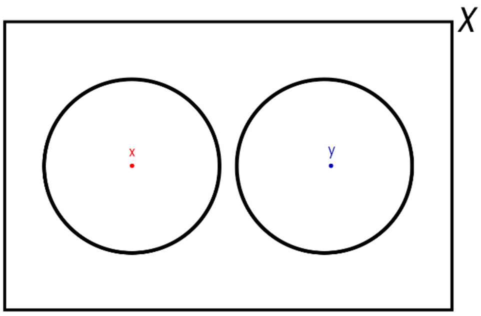

Fig. 9. Representation of two people in a room with a circle around them.

Now, if the circles around each person are to have a radius of 0.75 meters to 1 meter, then fig. 9 becomes;

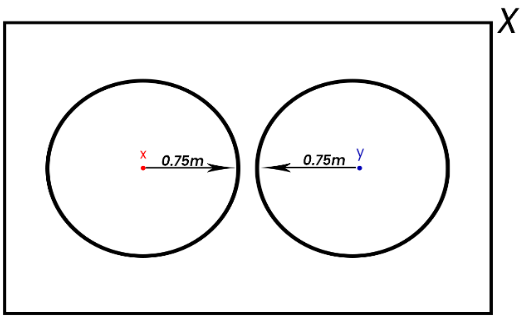

Fig. 10. Representation of 0.75m radius circle around persons in the room.

That is; x, y in X for $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$. Hence the Application of Hausdorff Space.

IV. Conclusions and Recommendations

Hausdorff space is a topological space with a separation property in that any two distinct points can be separated by a disjointed open set. This study succeeded in making Hausdorff Space real and outlined some of its applications. The study employed the bonding of inert gases and fully described the various types of bonding in the context of topology. Also, one application was seen in controlling the COVID-19 pandemic.

Two sets with the separation property are shown in Figure 6. This is a representation of the open sets U and V such that $x \in U$ and $y \in V$. Figure 7 depicts the representation of Hausdorff Space's reality by the inert gases Xenon and Radon. It was discovered that the two elements, Xenon and Radon, have nothing in common, implying that their intersection is empty, as illustrated in Figs. 9 and 10. To avoid students having the misconception that the subject is difficult, it is recommended that it be taught practically. Furthermore, it is widely advocated that some of the applications be stated to pique students' interest in this area, as well as those real-life applicable questions, be asked in addition to the theory.

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