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The Numerical Work Done by Transformations on a Symmetric Group

Rotimi Kehinde [1 *], D.I. Lanlege [2], A.I. Ma'ali [3], Abdulrahman Abdulganiyu [4] and Abdulazeez O. Habib [5]

[1] Department of Mathematical Sciences, Federal University Lokoja, Kogi State, Nigeria.
[2] Department of Mathematical Sciences, Federal University Lokoja, Kogi State, Nigeria.
[3] Department of Mathematical Sciences, Ibrahim Badamasi Babangida University Lapai, Niger State, Nigeria.
[4] Department of Computer Science, Ibrahim Badamasi Babangida University Lapai, Niger State, Nigeria.
[5] Department of Mathematical Sciences, Federal University Lokoja, Kogi State, Nigeria.
*Corresponding author email id: kennyrot2000@yahoo.com

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***Abstract* – The symmetric group of degree n (n is fixed) on a set is the group consisting of all bijections of the set (that is, all one-to-one and onto functions) from the set n = {1, 2, 3,…,n} to itself with function composition as the groups' operation. In this research, we have mainly presented the total work done, the average work done as well as the power of the transformations on the symmetric group of degree n by assuming elements of n to be set of points that are equally spaced on a line. We have used an effective methodology to generate and generalize formulas for calculating the total work done, the average work done and the power of the transformation by elements of the symmetric group. Results show that there is a huge amount of work done in moving elements of the domain to the elements of the co-domain of the fixed set. We also computed for small values of n the total work done, the average work done by elements of the symmetric group of transformation and the solutions obtained are illustrated graphically. This research work has an important application in physics and theoretical computer science.**

***Keywords* – Bijections, Total Work Done, Average Work Done, Power, Partial Transformation.**

I. INTRODUCTION

Transformation semigroup is one of the most fundamental mathematical objects. They arise naturally as endomorphism semigroup of various mathematical structures. Transformation semigroup is also of utmost importance for semigroup theory, as every semigroup is isomorphic to a transformation. The theory of finite semigroup have been of particular importance in theoretical computer science since 1950s because of the natural link between finite semigroup and finite automata via the syntactic monoids. Many researchers have worked and contributed immensely to the development of semigroup theory. At an algebraic conference in 1972 Schein surveyed the literature on semigroup of relations on n. In 1997, Schein and Ralph Mckenzie proved that every semigroup is isomorphic to a transitive semigroup of binary relations. In recent years many researchers in the field of abstract algebra have become more specialized with dedicated monographs appearing on important classes of semigroup as well as monographs focusing on applications in algebraic automata theory, particularly for finite automata and also functional analysis. The aim of this research work is to study and investigate all the transformations on a symmetric group of degree n and we will look at the numerical work done, the average work done by transformations on symmetric group of any given degree which is a subset of a partial transformation semigroup (PT_n). The power of the transformation measures numerically the strength of the transformation on a symmetric group for some given finite time in space. This power is wholly dependent on the total work done by transformations on the symmetric group of degree n. The symmetric group on a set of size n is the Galois group of the general polynomial of degree n and plays an important role in Galois theory. In invariant theory, the symmetric group acts on the variables of a multi-variate function and the functions left invariant are the so-called symmetric functions. In the representation theory of Lie groups, the representation theory of the symmetric group plays a fundamental role through the ideas of Schur functions. In the theory of Coxeter groups, the symmetric group is the Coxeter group of type A_n and occurs as the Weyl group of the general linear group. In combinatorics, the symmetric groups, their elements (permutations) and their representations provide a rich source of problems involving young tableaux, plactic monoids, and the Bruhat order. Subgroups of symmetric groups are called permutation groups and are widely studied because of their importance in understanding group actions, homogenous spaces and automorphism groups of graphs, such as the Higman-Sims group and the Higman–Sims graph. This research work is aimed at contributing to the advancement of group and semigroup theory and this we have achieved by providing a formula to measure and weigh the total work done, average work done and the power by transformations on any symmetric group of degree n. We have also provided analysis on the behavior of this quantities in the long run and the results of this research work can help in the understanding of computer languages using the structure and properties of the symmetric group. Laradji and Umar had proved and worked extensively on the combinatorial properties of semigroups of transformation, in their work they have provided a generalized cardinality of the symmetric group of degree n and this will form an important part of this research work as we have advanced the usage of this cardinality using an appropriate lemma. We are using the concepts of work done in physics to come up with generalized ideas. The results obtained has an important application in some branch of physics and in theoretical computer sciences.

II. Materials and Methods

If n is a fixed positive integer, the symmetric group of degree n is the set of all permutations on the fixed set n. This is denoted by S_n. That is, it is the set of all functions or transformations on a given set to itself, the operation of a symmetric group is **"function composition".** The symmetric group of degree n is a subset of the finite partial transformation semigroup of degree $n(PT_n)$, i.e. $S_n \subseteq PT_n$.

Supposed $n = \{1, 2, 3, 4, \cdots, n\}$, then the finite partial transformation semigroup PT_n, is the set of all partial transformations on n, that is, all functions between subsets of n including the void set and n itself. There are many of these subsets but our main goal is to consider the numerical work done by a transformation on symmetric group of any given degree which is a subset of PT_n. Let us consider the elements of n as set of points that are equally spaced on a line as shown below:

If τ is a transformation on S_n, it moves an element in the domain of n to an element in the co-domain of n by a distance given by $d(x,y) = |x - y|$. The total work done on S_n will be the sum of these distances as τ varies over the domain of n. This total work done will be denoted by $W_\tau(S_n)$. Consequently the average work done denoted $\overline{W}_\tau(S_n)$ by elements of S_n and the power of the transformations denoted by $P_t(S_n)$ can be calculated using the following relations: $\overline{W}_\tau(S_n)=\frac{W_\tau(S_n)}{|S_n|}$ and $P_t(S_n)=\frac{W_\tau(S_n)}{t}$, $t > 0$ respectively. Where $|S_n|$ is the order of S_n.

Consider the symmetric group of degree 3 (S_3) for clarity, attempt will be made to compute numerically the total work done, the average work done and the power for some given time in space by all transformations τ on

S_n all in terms of the distance function $d(x,y) = |x - \tau(x)|$. Below are the possible functions or transformation on S_3.

$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $\quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

$\tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $\quad \tau_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\tau_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $\tau_6 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

It is obvious that $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$ and τ_6 are all transformations on S_3 and τ_1 is a special transformation because it fixes elements of S_3, that is $\tau_1(1) = 1, \tau_1(2) = 2$ *and* $\tau_1(3) = 3$. We will now compute numerically the work done by each of these transformations using the definition defined earlier on a distance function, i.e. $d(x,y) = |x - \tau(x)|$.

For τ_1 we have

$W_{\tau_1}(S_3) = |1-1| + |2-2| + |3-3| = 0$

For τ_2 we have

$W_{\tau_2}(S_3) = |1-2| + |2-3| + |3-31| = 4$

For τ_3 we have

$W_{\tau_3}(S_3) = |1-1| + |2-3| + |3-2| = 2$

For τ_4 we have

$W_{\tau_4}(S_3)\ |1-3| + |2-1| + |3-2| = 4$

For τ_5 we have

$W_{\tau_5}(S_3)\ |1-3| + |2-2| + |3-1| = 4$

And for τ_6 we have

$W_{\tau_6}(S_3)\ |1-2| + |2-1| + |3-3| = 2$

Now the total work done is obtained by summing up the work done by each of the transformations τ on S_3.That is:

$W_\tau(S_3) = $ =16unit.

$W_\tau(S_3) = 1.6 \times 10^1$

This follows immediately that the average work done is given by;

$\bar{W}_\tau(S_3) = \frac{W_\tau(S_3)}{|S_3|}$ But $|S_3| = 3! = 6$.

So $\bar{W}_\tau(S_3) = \frac{16}{6} = 2.67unit(3sf) = 2.67 \times 10^0$

And the power of τ for $t = 5$ on S_n is given by $P_5(S_3) = \frac{W_\tau(S_3)}{5}$

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Therefore $P_5(S_3) = \frac{16}{5} = 3.2unit$

Now we go ahead to generalize the intuitive concept to symmetric group of any degree n.

LEMMA

Let S be a subset of the partial transformation semigroup, that is, $S \subseteq PT_n$ then

$$W_\tau(S) = \sum_{x,y\epsilon n} |x - \tau(x)| \cdot n_{xy}(S). \tag{1}$$

Where n_{xy} is the cardinality of S.In the case of symmetric group of degree n, $S = S_n$, let us define S_n as given below:

$$S_n = \{\tau \epsilon T_n | \tau(x) = y\} \tag{2}$$

Which is the set of all functions which move element x to y. This immediately follows that $n_{xy}(S_n) = (n-1)!\,.\,n_{xy}$ [2] (A.Umar, 2014)

This is the cardinality of S_n, then

$$W_\tau(S_n) = \sum_{x,y\epsilon n} |x - \alpha(x)| \cdot n_{xy}(S_n). \tag{3}$$

Let us recall that for each $1 \le x \le y \le n$, we have:

$$\sum_{1 \le x \le y \le n} |x - \tau(x)| = \binom{n+1}{3} \tag{4}$$

This immediately follows that;

$\sum_{x,y\epsilon n} |x - y| = 2 \cdot \binom{n+1}{3}$, on expansion we get $2 \cdot \binom{n+1}{3} = 2 \cdot \frac{(n+1)!}{(n-2)!3!} = \frac{2[(n+1)\cdot n(n-1)(n-2)!}{(n-2)!\cdot 3!}$

$$\sum_{x,y\epsilon n} |x - y| = \frac{n(n^2-1)}{3} \tag{5}$$

Substituting equation (5) and the value of $n_{xy}(S_n)$ into equation (3), we obtain the total work done by element of S_n given below :

$$W_\tau(S_n) = \sum_{x,y\epsilon n} |x - \alpha(x)| \cdot n_{xy}(S_n) = \frac{(n^3-n)}{3} \cdot (n-1)! \, \frac{(n-1)[n(n+1)(n-1)(n-2)!}{3}$$

$$W_\tau(S_n) = \frac{(n-1)(n+1)!}{3} \tag{6}$$

Consequently, the average work done by element of S_n is given by:

$\overline{W}_\tau(S_n) = \frac{W_\tau(S_n)}{|S_n|} = \frac{(n-1)(n+1)!}{3\cdot n!}$, since $|S_n| = n!$

$$\overline{W}_\tau(S_n) = \frac{n^2-1}{3} \tag{7}$$

And the generalized power of τ on S_n for any given time in space is;

$$P_t(S_n) = \frac{W_\tau(S_n))}{t}$$

$$P_t(S_n) = \frac{(n-1)(n+1)!}{3t} \quad t > 0 \tag{8}$$

III. Results and Discussions

In this section we now calculate and tabulate using the formulas obtained above for $W_\tau(S_n), \overline{W}_\tau(S_n)$ and $P_t(S_n)$ using small values of $n \leq 10$, where $n \in \mathbb{N}$ and a time is space of $t = 5units$. The results are shown in the tables below:

Table 1. Results Obtained for the total work done and the Average Work done by Elements of S_n.

n	$W_\tau(S_n)$	$\overline{W}_\tau(S_n)$
1	0.00×10^0	0.00×10^0
2	2.00×10^0	1.00×10^0
3	1.60×10^1	2.67×10^0
4	1.20×10^2	5.00×10^0
5	9.60×10^2	8.00×10^0
6	8.40×10^3	1.70×10^1
7	8.06×10^4	1.60×10^1
8	8.47×10^5	2.10×10^1
9	9.68×10^6	2.67×10^1
10	1.20×10^8	3.30×10^1

Table 2. Corresponding powers of S_n for the given time in space.

$P_5(S_1)$	0.00×10^0
$P_5(S_2)$	4.00×10^{-1}
$P_5(S_3)$	3.20×10^0
$P_5(S_4)$	2.40×10^1
$P_5(S_5)$	1.92×10^2
$P_5(S_6)$	1.68×10^3
$P_5(S_7)$	1.61×10^4
$P_5(S_8)$	1.69×10^5
$P_5(S_9)$	1.94×10^6
$P_5(S_{10})$	2.40×10^7

We will now illustrate the results of Table 1 above on a graph as shown in Fig 1 and Fig 2 below:

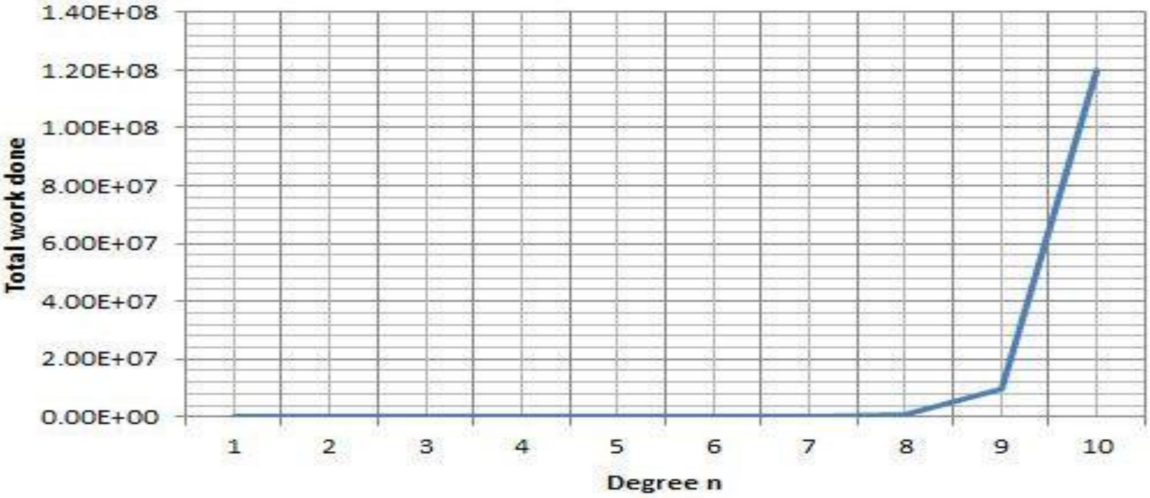

Fig. 1. Graph of the total work done by elements of S_n.

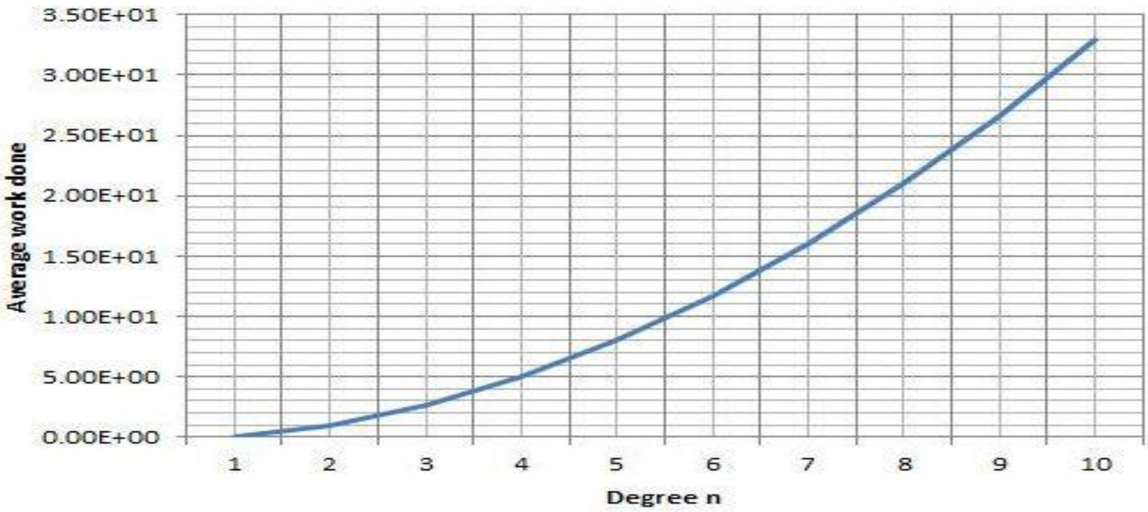

Fig. 2. Graph of the average work done by elements of S_n.

It is clear from Table 1 above that the values of $W_\tau(S_n)$ and $\bar{W}_\tau(S_n)$ approached ∞ as $n \to \infty$, this shows numerically the work performed in moving a point from domain to co-domain by functions that are elements of S_n and this work done is dependent of the degree n of the symmetric group, that is the higher the degree n, the higher the amount of work required to move the elements and this is consistent with the nature of the illustrative graphs provided. We also see that the formulas derived with the aid of the lemma and the cardinality of S_n is highly consistent with the intuitive analysis that was initially given for S_3, that is; substituting $n = 3$ into the formulas for total work done and the average work done as shown in the Table 1 above gave the same results as to the intuitive solution and this further confirms the validity and the reliability of these formulas. The power of the transformation is just the ratio of the total work done to a fixed given time t and this tells us the strength of the function τ on a numerical point of view, this power is also dependent on the degree of the finite symmetric group of transformation.

IV. Conclusion

In this research, we have successfully computed the numerical work done by a subset of a partial transformation semigroup PT_n, specifically $\mathrm{S_n}$ and we have also computed the average work done and the power of the transformation on S_n and from our results we found that $W_\tau(S_n)$, $\overline{W}_\tau(S_n)$ and $P_t(S_n)$ grow without bound and this was consistent with nature of the graph above. We have also derived a generalized formula to compute $W_\tau(S_n)$, $\overline{W}_\tau(S_n)$ and $P_t(S_n)$ for a symmetric group of any degree n and this was consistent with the intuitive analysis we gave for S_3. When $n = 1$, we see from the table that $W_\tau(S_n) = \overline{W}_\tau(S_n) = 0$ and this literarily means that there is no work done by the transformation. The numerical values generated above can be applied in theoretical computer science where the properties of the computer language depends on the algebraic properties of the symmetric group. We shall be looking at this application of the structure of S_n and its transformations to computer languages in our future research.

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Author's Profile

First Author
R. Kehinde is a Doctor of Philosophy (PhD) in mathematics (Algebra option). He is currently a senior lecturer at Federal University Lokoja, Kogi State Nigeria. He has contributed immensely to the development of mathematical knowledge and has several of his paper cited by various academicians according to Google scholar. He has published up to 28 academic papers both in local journals, university based journals and international journals.

Second Author
D.I. Lanlege is a Doctor of Philosophy (PhD) in mathematics. He is currently a senior lecturer at Federal University Lokoja, Kogi State Nigeria. He has contributed greatly to the development of mathematical knowledge especially in the area of computational mathematics, differential equations and mathematical physics. He has published more than 36 academic papers both in local journals and international journals. email id: loislanlege@yahoo.com

Third Author
Doctor Aliyu Ishaku Ma'ali was born on 1st March 1971 in Ma'ali village via Batati in Lavun Local government Area of Niger State, Nigeria. He received his elementary education in Nigeria. He then proceeded to Ahmadu Bello University Zaria, Nigeria where he obtained diploma in mathematics education. He equally proceeded by gaining admission to same University where he now obtained Bachelor of Science education in mathematics. He proceeded to University of Ilorin where he obtained masters degree in applied mathematics as well as Doctor of Philosophy in mathematics (PhD). He has been a mathematics lecturer at Ibrahim Badamasi Babangida University, Lapai, Niger State, Nigeria since 2005, he held various position including being the current head of department of Mathematics. He has more than 40 academic papers published in both local and international Journals. email id: amaalii@ibbu.edu.ng

Fourth Author
Abdulrahman Abdulganiyu is a lecturer 1 in the department of Computer Science, Faculty of natural Sciences, Ibrahim Badamasi Babangida University Lapai, Niger State, Nigeria. He holds a BSc in computer science from the university of Abuja, Abuja, Nigeria, a master degree in Computer Science from the university of Ibadan and he is currently running his PhD in computer science at University of Abuja. Abdulganiyu is a specialist in information and cyber security, he is a member of the computer professional of Nigeria (CPN), the society of Digital Information and wireless communications (SDIWC) and the internet society (ISOC). His scholarly publication has appeared and has been quoted in reputable peer-refereed Journals, Conference proceedings, Newsletters and edited books. email id: abdulg2009@yahoo.com

Fifth Author
Abdulazeez Onimisi Habib was born on 22nd November 1993. He obtained his first school leaving certificate and secondary school certificate in Nigeria. He is currently a final year student studying at Federal University Lokoja, Kogi state Nigeria as an undergraduate in the department of Mathematical sciences and he has great interest in pure mathematics. email id: abdulazeezhabibonimisi@gmail.com