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# Decompositions of the Complete Mixed Graph into Mixed Stars

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***Abstract*** **– In the widely studied area of graph and digraph decompositions, little attention has been given to mixed graph decompositions. In this paper, necessary and sufficient conditions are given for the isomorphic decomposition of the complete mixed graph into each of the four partial orientations of a 6-star which has two edges and four arcs, following up on a similar result for partial orientations of a 3-star given by Beeler and Meadows (in "Decompositions of mixed graphs using partial orientations of $P_4$ and $S_3$," *International Journal of Pure and Applied Mathematics*, Volume 56, Issue 1, pp. 63-67, 2009). Some results concerning decompositions of the complete graph into partial orientations of other stars are also given.**

***Keywords*** **– Graph Decompositions, Mixed Graphs, Orientations of Stars.**

## I. INTRODUCTION

A *$g$-decomposition of graph $G$* is a set of subgraphs of $G$, $\gamma = \{g_1, g_2, \ldots, g_n\}$, where $g_i \cong g$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} E(g_i) = E(G)$. The $g_i$ are called *blocks* of the decompositi--on. When $G$ is a complete graph, the $g$-decomposition is often called a *graph design*. The literature on graph designs and graph decompositions is extensive [4, 10]. However, decompositions of mixed graphs seem little-studied.

A mixed graph consists of a set $V$ of vertices, a set $E$ of edges (which correspond to unordered pairs of distinct vertices) and a set $A$ of arcs (which correspond to ordered pairs of distinct vertices). Mixed graphs often arise in the study of network flows (see Chapter 9 of [1]). An application concerning mixed graph decompositions can be found in [9]. The *complete mixed graph* of order $v$, $M_v$, has a vertex set of cardinality $v$, edge set $E = \{uw \mid u \in V, w \in V, \text{and } u \neq w\}$, and arc set $A = \{(u, w) \mid u \in V, w \in V, \text{and } u \neq w\}$. So between any two distinct vertices of $M_v$ are one edge and two arcs (the arcsare converses of each other). Therefore $M_v$ has twice as many arcs as edges and a necessary condition for the existence of an $m$-decomposition of $M_v$ is that $m$ has twice as many arcs and edges. Decompositions of $M_v$ were first addressed in the setting of mixed triple systems, where necessary and sufficient conditions were given for decompositions of $M_v$ into each of the three partial orientations of a 3-cycle with two arcs and one edge [7]. Decompositions of $M_v$ into partial orientations of 3-paths and 3-stars (each with two arcs and one edge) were explored in [2]; in fact, necessary and sufficient conditions were given for such decompositions of the $\lambda$-fold complete graph, $\lambda M_v$. More recently, the decomposition of certain mixed graphs was applied to the conjecture of Favaron et al. [6] concerning the $2k+1$-path decomposition of a $2k+1$-regular graph with a perfect matching [5]. The purpose of this paper is to address decompositions of $M_v$ into copies of partial orientations of the 6-star which have two edges and four arcs.

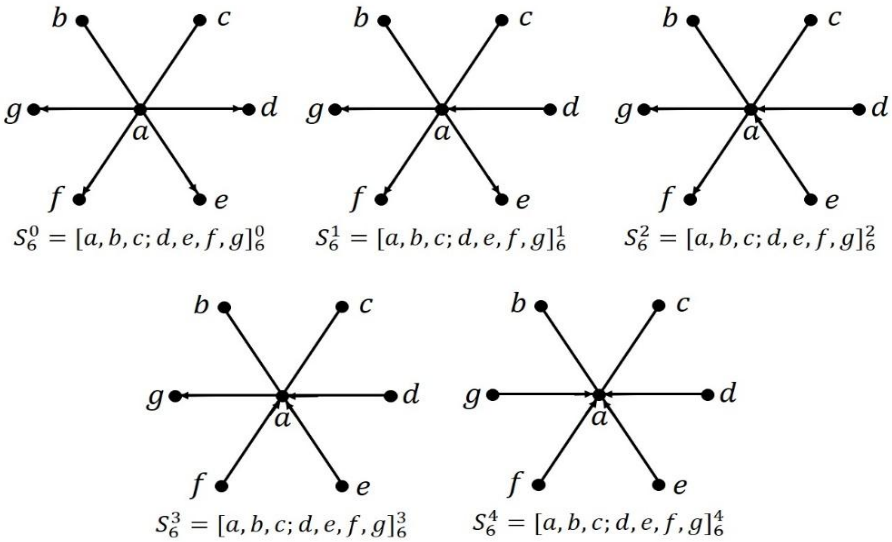


Fig. 1. The five partial orientations of the 6-star which have two edges and four arcs.

## II. Results for Partial Orientations of 6-Stars

We now consider partial orientations of a 6-star which has two edges and four arcs. In Figure 1 we give drawings of these mixed stars and introduce notation for the representation of each of the five such mixed stars. Notice that we denote the partial orientation of the 6-star with two edges, four arcs, and with the center of the star having in-degree $i$ and out-degree $4 - i$ as $S_6^i$. We now give necessary and sufficient conditions for the existence of an $S_6^i$- decomposition of $M_v$ for each of $i = 1, 2, 3, 4$. In each case, we give a direct construction to establish sufficiency.

1. *Theorem*

A $S_6^0$-decomposition of $M_v$ exists if and only if $v \equiv 1$ (mod 4), $v \geq 9$.

*Proof.*

Every vertex of $S_6^0$ is of out-degree 4 and every vertex of $M_v$ is of out-degree $v - 1$, so $v \equiv 1$ (mod 4) is a necessary condition for such a decomposition. Since $S_6^0$ has 7 vertices, then $v \geq 9$ is also necessary.

For sufficiency, let $v = 4k + 1$ where $k \geq 2$ and let the vertex set of $M_v$ be $\{0, 1, 2, \ldots, 4k\}$. Consider the set $B = \{[0, 4k - 1, 4k; 1, 2, 3, 4]_6^0\} \cup \{[0, 3 + 2j, 4 + 2j; 5 + 4j; 6 + 4j; 7 + 4j; 8 + 4j]_6^0 \mid j = 0, 1, 2, \ldots, k - 2\}$.

The copies of $S_6^0$ in set $B$, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 4k)$ form a $S_6^0$-decomposition of $M_v$, as claimed. ∎

Notice that the converse of $S_6^0$ (that is, the mixed graph that results by reversing the direction of each arc) is $S_6^4$, and $M_v$ is self-converse, so theorem 1 also gives the necessary and sufficient conditions for a $S_6^4$-decomposition of $M_v$.

1. *Lemma*

For $i \in \{1, 2, 3\}$, a$S_6^i$-decomposition of $M_v$ does not exist when $v \equiv 2$ or 3 (mod 4).

*Proof.*

With $v \equiv 2$ or 3 (mod 4), the number of arcs in $M_v$ is $v(v-1) \equiv 2$ (mod 4), but since $S_6^i$ has 4 arcs, then a necessary condition for the existence of a $S_6^i$-decomposition of $M_v$ is that the number of arcs in $M_v$ is 0 (mod 4). The claim follows. ■

2. *Lemma*

For $i \in \{1,2,3\}$, a $S_6^i$-decomposition of $M_8$ does not exist.

*Proof.*

First, $M_8$ has 28 edges and $S_6^i$ has 2 edges, so a$S_6^i$-decomposition of $M_8$ requires 14 copies of $S_6^i$. Now two vertices $a$ and $b$ are ends of an edge of $S_6^i$ if and only if either $a$ or $b$ is the center of $S_6^i$. So for the edge $ab$, the arc $(a, b)$, and the arc $(b, a)$ to appear in a $S_6^i$-decomposition of $M_8$ we need the number of times $a$ is the center in some $S_6^i$ plus the number of times $b$ is the center in some $S_6^i$ to be at least three. But $M_8$ has 8 vertices and a$S_6^i$- decomposition of $M_8$ requires 14 copies of $S_6^i$, so at least two vertices, say vertices $c$ and $d$, are the center of at most one copy of $S_6^i$ and then the copies of $S_6^i$ cannot include each of edge $cd$, arc $(c, d)$, and arc $(d, c)$. That is, a$S_6^i$- decomposition of $M_8$ does not exist. ■

2. *Theorem*

A $S_6^1$-decomposition of $M_v$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \geq 9$.

*Proof.*

The necessary condition follows from Lemmas 1 and 2.

For sufficiency, first let $v \equiv 0$ (mod 4), say $v = 4k$ where $k \geq 3$ and let the vertex set of $M_v$ be $\{0, 1, 2, \dots, 4k-2, \infty\}$. Consider the set $B = \{[0, \infty, 1; 4k - 2, 2, 3, 4]_6^1, [0, 2, 3; \infty, 5, 6, 7]_6^1, [0, 4, 5; 1, 8, 9, \infty]_6^1,\} \cup \{[0, 6 + j, 2k - 1 - j; 2 + 4j; 10 + 4j; 11 + 4j; 12 + 4k]_6^0 \mid j = 0, 1, 2, \dots, k - 4\}$.

The copies of $S_6^1$ in set $B$, along with their images under the powers of the permutation $(\infty)(0, 1, 2, \dots, 4k - 2)$, form a $S_6^1$-decomposition of $M_v$, as claimed.

Next let $v \equiv 1$ (mod 4), say $v = 4k + 1$ where $k \geq 2$ and let the vertex set of $M_v$ be $\{0, 1, 2, \dots, 4k\}$. Consider the set $B = \{[0, 1, 4k - 1; 4k, 2, 3, 4]_6^1\} \cup \{[0, 3 + 2j, 4 + 2j; 1 + j, 5 + 3j, 6 + 3j, 7 + 3j]_6^1 \mid j = 0, 1, 2, \dots, k - 2\}$.

The copies of $S_6^1$ in set $B$, along with their images under the powers of the permutation $(0, 1, 2, \dots, 4k)$, form a $S_6^1$- decomposition of $M_v$, as claimed. ■

The converse of $S_6^1$ is $S_6^3$, and $M_v$ is self-converse, so theorem 2 also gives the necessary and sufficient conditions for a $S_6^3$- decomposition of $M_v$.

3. *Theorem*

A $S_6^2$-decomposition of $M_v$ exists if and only if $v \equiv 0$ or 1 (mod 4), $v \geq 9$.

*Proof.*

The necessary condition follows from Lemmas 1 and 2.

For sufficiency, first let $v \equiv 0 \pmod 4$, say $v = 4k$ where $k \geq 3$ and let the vertex set of $M_v$ be $\{0, 1, 2, \ldots, 4k-2, \infty\}$. Consider the set $B = \{[0, \infty, 1; 2, 4k-2, 3, 4]_6^2, [0, 2, 3; \infty, 1, 5, 6]_6^2, [0, 4, 5; 4k-8, 4k-9, \infty, 2]_6^2,\} \cup \{[0, 6+2j, 7+2j; 3+2j, 4+2j, 9+2j, 10+2j]_6^2 \mid j = 0, 1, 2, \ldots, k-4\}$.

The copies of $S_6^2$ in set $B$, along with their images under the powers of the permutation $(\infty)(0, 1, 2, \ldots, 4k-2)$, form a $S_6^2$-decomposition of $M_v$, as claimed.

Next let $v \equiv 1 \pmod 4$, say $v = 4k+1$ where $k \geq 2$ and let the vertex set of $M_v$ be $\{0, 1, 2, \ldots, 4k\}$. Consider the set $B = \{[0, 4k-1, 4k; 2, 3, 1, 4]_6^2\} \cup \{[0, 3+2j, 4+2j; 6+4j, 7+4j, 5+4j, 8+4j]_6^2 \mid j = 0, 1, 2, \ldots, k-2\}$.

The copies of $S_6^2$ in set $B$, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 4k)$, form a $S_6^2$-decomposition of $M_v$, as claimed. ■

The results of the section combine to give us the necessary and sufficient conditions for an $S_6^i$-decomposition of $M_v$ for $i = 0, 1, 2, 3, 4$, as follows.

4. *Theorem*

A $S_6^i$-decomposition of $M_v$ exists if and only if $v \geq 9$ and

1. If $i \in \{0, 4\}$ then $v \equiv 1 \pmod 4$, and
2. If $i \in \{1, 2, 3\}$ then $v \equiv 0$ or $1 \pmod 4$.

## III. Some Results for Partial Orientations of General Stars

We now give two results concerning decompositions of $M_v$ involving partial orientations of a star $S_{3k}$ (where $k \in \mathbb{N}$), where it is necessary that the partial orientation has $k$ edges and $2k$ arcs. We denote by $S_{3k}^i = [v_0, v_1, v_2, \ldots, v_k;\ v_{k+1}, v_{k+2}, \ldots, v_{3k}]_{3k}^i$ the orientation of $S_{3k}$ with edge set $\{v_0v_1, v_0v_2, \ldots, v_0v_k\}$ and arc set $\{(v_{k+1}, v_0), (v_{k+2}, v_0), \ldots, (v_{k+i}, v_0), (v_0, v_{k+1+i}), (v_0, v_{k+2+i}), \ldots, (v_0, v_{3k})\}$. We give direct constructions and, as in the previous section, we take advantage of a cyclic permutation. We do not give necessary and sufficient conditions, but instead show sufficiency for some specific values of $v$.

5. *Theorem*

A $S_{3k}^i$-decomposition of $M_v$ exists for $v = 4k+1$ and for each $i = 0, 1, 2, \ldots, 2k$.

*Proof.*

Let the vertex set of $M_v$ be $\{0, 1, 2, \ldots, 4k\}$. Consider the set $B = \{[0, 4k, 4k-1, \ldots, 3k+1; 1, 2, 3, \ldots, 2k]_{3k}^i, [0, 2k, 2k-1, \ldots, k+1; 4k, 4k-1, \ldots, 2k+1]_{3k}^i\}$.

The copies of $S_{3k}^i$ in set $B$, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 4k)$, form a $S_{3k}^i$-decomposition of $M_v$, as claimed. ■

6. *Theorem*

A $S_{6k+3}^i$- decomposition of $M_v$ exists for $v = 12k+7$ and for each $i = 0, 2, 4, \ldots, 4k+2$.

*Proof.*

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Let the vertex set of $M_v$ be $\{0, 1, 2, \ldots, 12k + 6\}$. Consider the set $B$ = {[0, 12$k$ + 6, 12$k$ + 5,…,10$k$ + 6; 1, 2,…,4$k$ + 2]$^i_{6k+3}$, [0, 8$k$ + 4, 8$k$ + 3,…,6$k$ + 4; 12$k$ + 6, 12$k$ + 5,…,8$k$ + 5]$^i_{6k+3}$, [0, 10$k$ + 5, 10$k$ + 4,…,8$k$ + 5; 8$k$ + 4, 8$k$ + 3,…,8$k$ + 5 – $i$ / 2, 4$k$ + 3, 4$k$ + 4,…,,8$k$ + 4 – $i$ / 2]$^i_{6k+3}$}.

The copies of $S^i_{3k}$ in set $B$, along with their images under the powers of the permutation $(0, 1, 2, \ldots, 12k + 6)$, form a $S^i_{6k+3}$- decomposition of $M_v$ where $i = 0, 2, 4, \ldots, 4k + 2$, as claimed. ■

We see from these two results that there is much potential for additional research concerning the existence of $S^i_{3k}$- decompositions of $M_v$.

## IV. Conclusion

Theorem 4 gives necessary and sufficient conditions for the decomposition of the complete mixed graph into copies of$S^i_6$ for $i = 0, 1, 2, 3, 4$. This is a second step (following the first step taken by Beeler and Meadows for partial orientations of an $S_3$in [2]) in the general problem of the existence of $S^i_{3k}$- decompositions of $M_v$. Cyclic permutations were used in the constructions of Theorems 1, 2, and 3 in the cases when $v$ was odd. The constructions for $v$ even employed a permutation with one fixed point and cycle of length $v - 1$; such a permutation is called rotational or 1-rotational [11]. Star decompositions of the complete graph which admit cyclic or rotational automorphisms are explored in [8], and rotational mixed triple systems (as well as mixed triple systems admitting bicyclic and reverse automorphisms) are addressed in [3]. Future research, in addition to the existence problem, could focus on $S^i_{3k}$ -decompositions of $M_v$ which admit cyclic, rotational, or other types of automorphisms.

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