International Journal of Innovation in Science and Mathematics
Volume 7, Issue 2, ISSN (Online): 2347–9051

Hierarchical Smoothing of Residues Method in Solving Elliptic Problems

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Date of publication (dd/mm/yyyy): 26/03/2019

Abstract **– The aim of the current article is to provide an alternative numerically method in solving elliptic problems.**

The method is called hierarchical smoothing of residues HSRM.

It is based on iteratively smoothing of the residual on the same grid through a smoothing operator on the same grid this reduce the complexity compare to the multigrid method.

The Fourier method has been used for the convergence analysis.

Numerical tests done are very promising for its efficiency.

Keywords **– Multi-Level Smoothing, Residue-Fourier Analysis, Multigrid–Finite, Difference-Finite, Element-Runge Kutta, Euler Method.**

I. Introduction

We present a numerical method in solving elliptic linear and nonlinear problems.

We consider in the first step the Hilbertian case.

Let us consider a general elliptic operator L from the Hilbert spaces H and V.

V is imbedded continuously into H.

We consider the elliptic operator L linear or not from V to V' the dual space of V.

(1.1) $-Lu = f$ *in the Weak Sense*

We assume the problem well posed.

The associate discretized equation in an appropriate space can be written.

(1.2) (P_h) $-L_h u_h = f_h$

U_h in V_h et f_h in H_h finite subspaces of V and H respectively.

We consider in the first step the linear case and use the matrix representation of the discrete operators.

II. A class of Iterative Method

We consider a general class of iterative methods define by the operators :

$$N_h = I_h - B_{h.}.L_h \quad (2.1)$$

B_h is an operator that characterizes the iterative method and is in such that:

$$\|I_h - B_h.L_h\| < 1 \quad (2.2)$$

Condition that ensures the convergence of the method.

Many iterative methods can be represented in this form.

We define here the stationnarisation method that can be put into the form (2.1) and linked to evolution equations.

III. Iterative Methods and Evolution Equations

We consider the parabolic equation well posed

$$\frac{du}{dt} = Lu + f(t) \;\; t > 0 \tag{3.1}$$

$$u(0) = u_0 \tag{3.2}$$

The method of stationnarisation consists of constructing the stationary solution of (1.1) as limit of the solutions of (3.1) independent of the initial conditions.

The procedure of so obtaining the stationary solutions is called stationnarisation.

The stationnarisation is carried out through some numerical methods.

We propose and analyse below some alternative method.

IV. Iterative Schemes for the Stationnarisation

4.1 *Hierarchical Smoothing of Residues*

Let us consider the discrete equation

$$L_h u_h = f_h \; on \; the \; grid \; \omega_h \qquad (4\text{-}1)$$

Many methods can be used to solve (4-1):

Let u_h^j be an approximation of the solution of u_h and $v_h^j = u_h - u_h^j$ *the error at the step* j

The residue is $d_h^j = f_h - L_h u_h^j$

The residual equation reads :

$$L_h v_h^j = d_h^j \qquad (4\text{-}2)$$

The equations (4-1) and (4-2) are of the same form.

$$u_h = u_h^j + v_h^j$$

A new approximation of u_h is obtained from any approximation $\hat{v}_h^j$ *of* v_h^j and we can write $u_h^{j+1} = u_h^j + \hat{v}_h^j$

The approximation $\hat{v}_h^j$ of the solution of (4-2) is obtained through an approximation of L_h^{-1} denoted B_h and we can write (4-3) $v_h^{j+1} = (I_h - B_h L_h) v_h^j$

The classical condition of convergence is $\rho(N_h) < 1, \rho(C)$ *being the spectral radius of* C

The costs of the iterative methods or any multi-scale between which the multigrid method depend on the choice of B_h.

The idea of the multi-grid method consist of choosing a very cheap operator B_h on a macro-grid and by the use of the appropriate prolongation and restriction operators.

4.2. *Stationnarisation schemes*

Let us propose some schemes to be used in the construction of our method

ALGO1

Let V^m be an approximation of the exact solution at time $t = m\tau$, τ is the time step.

$$\frac{v^{m+1} - v^m}{\tau} + L_h v^m = f^m \quad m = 0,1 \dots \dots \qquad (4\text{-}2\text{-}1)$$

This is the explicit Euler method.

ALGO2

We can combine the fractional step and alternate direction to obtain.

$$\frac{v^{m+1} - v^{m+1/2}}{2\tau} + L^1_{h_h} v^m + L^2_h v^{m+1/2} = f^{m+1} \quad m = 0,1 \dots \dots \qquad (4\text{-}2\text{-}2)$$

$$\frac{v^{m+1/2} - v^m}{2\tau} + L^1_{h_h} v^{m+1/2} + L^2_h v^m = f^{m+1/2} \quad m = 0,1 \dots \dots \qquad (4\text{-}2\text{-}3)$$

$v^{m+1/2}$ is an intermediate approximation of v^{m+1}.

L^1_h *and* L^2_h are two operators in the decomposition of L_h Remark 4-2.

We choose in our tests the operators

L = -Δ the laplacian and L = (-Δ, ∇) the stokes operator L_h being the associate discrete operator by finite elements or finite difference or any numerical method.

The Von Neumann stability condition is given by $\left|\frac{\tau}{h^2}\right| < 1$ *for ALGO1*.

This does not enable large values of time step for small value of h.

The scheme ALGO2 is semi-implicit unconditional stable.

The stationary solution can be obtained in a very cheaper way.

We are presenting now our methods:

We choose a smoothing operator S.

4.3. *Euler Smoothing Scheme*

Let $(4-3-1)\sigma = \frac{\tau}{h^2}$ *and* $H = h^2 L_h$

We define the smoother

SALGO1N1

$$v^{n+1} = v^n + \sigma SOH(v^n) \qquad (4\text{-}3\text{-}2)$$

SALGO2N1

$$\frac{v^{m+1} - v^{m+1/2}}{\tau/2} + SOL^1_{h_h} v^m + L^2_h v^{m+1/2} = f^{m+1} \quad m = 0,1 \dots \dots \qquad (4\text{-}3\text{-}3)$$

$$\frac{v^{m+1} - v^{m+1/2}}{\tau/2} + L^1_{h_h} v^m + SOL^2_h v^{m+1/2} = f^{m+1} \quad m = 0,1 \dots \dots \qquad (4\text{-}3\text{-}4)$$

NB: In the algorithms (4-3-3) and (4-3-4) the smoothing is done in one direction and to the other

The smoothing is done once.

The smoothing can be done k-time on the same grid using the same smoothing operator S , in that case S is replaced by S^k.

The scheme obtained is called k-level.

We can define The Euler 2-level scheme

SALGO1N2

$$v^{n+1} = v^n + \sigma S^2 oL(v^n)$$

SALGO1Nk

$$v^{n+1} = v^n + \sigma S^k \, oL(v^n)$$

Similar schemes can be done for the ALGO2

4.4. *Richardson algorithms*

Using the results in [3] we can increase the performance of the Euler iterations in 4-3 by choosing a set of parameters σ_n the optimal value is not known in advance but is going to be estimated.

We can define the following Richardson schemes :

(4-4-1) $v^{n+1} = v^n + \sigma_n L_k(v^n)$ L_k being a k-level smoother operator.

The main difficulty is the determination of the optimal parameter.

If L is positive definite and the eigen values are in the interval (a,b) the optimal values are given by :

$$\sigma_i = \frac{\tau}{(a-b)t_i - (a+b)} \; where \; t_i = cos\left(\frac{(2i-1)\pi}{2n}\right), i = 1,2 \ldots \ldots n$$

To present the importance of our method, we outline the main procedure of the multigrid method for easier comparison.

4.6. *Multigrid Procedure*

The problem is to solve the discrete equation $L_h(u_h) = f_h$

Case of Two-grid (h, H)

H is the finest grid and H the macro

Let u_h^j an approximation of the exact solution u_h

The corresponding residue is defined by

i) $e_h^j = f_h - L_h(u_h^j)$

ii) Restriction on the grid H

$$e_H^j = I_h^H e_h^j$$

iii) Solve on the macro grid H

$$L_H(\widehat{v_H^j}) = e_H^j$$

iv) Interpolation on the fine grid

$$\widehat{v_H^j} = I_H^h \widehat{v_h^j}$$

v) Evaluation of the new approximation

$$u_h^{j+1} = u_h^j + \widehat{v_h^j}$$

The operators I_H^h and I_h^H are very difficult to manipulate.

Our method take into account the multigrid complexity.

4.7. *Hierarchical Smoothing Algorithms*

Let u_h^j be an approximation of u_h

i) Evaluate the residue

$$e_h^j = f_h - L_h(u_h^j) \tag{4-7-1}$$

ii) Smoothing the Residue

$$e_h^{j+1} = e_h^j + SoL_h(e_h^j) \tag{4-7-2}$$

S is the smoothing operator

iii) Approximation of order j+1

$$u_h^{j+1} = u_h^j + e_h^{j+1}$$

4.8. *Construction of the Smoothing Operator S*

S is chosen as simple as possible in such a way that

$$|1 + \lambda_S . \lambda_{L_h}| < 1$$

Where λ_S, λ_{L_h} are eigenvalues of S and L_h respectively

For the finite element or finite difference methods we choose S in block form

$$S = \begin{pmatrix} S_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_k \end{pmatrix}$$

In the same form of finite elements matrices

$$A = \begin{pmatrix} A_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_k \end{pmatrix}$$

In such a way that $S^m oA$ is *a* block matrice

With components $S_i oA_i$ easier to tackle.

V. Analysis of Some Model Problem in 2 Dimensional Space

Let us consider the Dirichlet problems

$$-\Delta u = f \ in \ \ \Omega \,, \ ^{(5-2)} \ u/_{\delta\Omega} = 0 \tag{5-1}$$

Where Ω is the unit square $[0,1]^2$

We can consider the Stokes equations in the same domain

We discretise the equation by the P1 conform uniform triangulation.

We obtain the following system :

$$-u_{i-1j}-u_{i,j-1}+4u_{i,j}-u_{i,j+1}-u_{i+1,j}=h^2 f_{i,j} \quad (5\text{-}3)$$

$$u_{i,j}=0 \text{ on the boundary} \quad (5\text{-}4)$$

We can note that the discretisation of 5-1 by the 5-points finite difference gives the same equations as in 5-3.

Let L be the linear operator associated to 5-3.

Let $(5-5)\ e_{ij}=e^{i\left(\theta_i^1+\theta_j^2\right)}\ \theta_k^p=\frac{k\pi}{N}, k=1,2\ldots\ldots N.$

The Fourier mode. We have by simple calculation $L\,e_{ij}=\left(-2cos\theta_i^1-2cos\theta_j^2+4\right)e_{ij}$

We can deduce the following

Lemma 5-1

The mode e_{ij} are eigenvectors of L associate to $\lambda_{ij}=\left(-2cos\theta_i^1-2cos\theta_j^2+4\right)$

5-8 *Construction of the Smoothing Operator S*

Let us define

$$S(e_{ij})=\frac{e_{i-1j}+e_{ij-1}+4e_{ij}+e_{i+1j}+e_{ij}}{8}$$

Where (I,j) is an interior point.

$$S(e_{0j})=\frac{e_{N-1j}+e_{0j-1}+4e_{0j}+e_{1j}+e_{0j}}{8}$$

$$S(e_{i,N})=\frac{e_{i-1N}+e_{iN-1}+4e_{iN}+e_{i1}}{8}$$

$$S(e_{N,j})=\frac{e_{N-1j}+e_{Nj-1}+4e_{Nj}+e_{i+1j+e_{Nj+1}}}{8}$$

We can note that S is a symmetric operator hence has a real spectrum. It then follows by simple calculation the following lemmata

Lemma 5-2

The modes $e_{ij}=e^{i\left(\theta_i^1+\theta_j^2\right)}$ are eigenvectors associated to the eigenvalues $\lambda_S=\frac{1}{2}\left(cos((\theta_i^1+\theta_j^2)/2)cos((\theta_i^1-\theta_j^2)/2)+1\right)$.

Lemma 5-3

The spectrum of the operator $L_1=I-\sigma SoL$.

Is $\lambda_{L1}=1-\sigma\lambda_S.\lambda_L$

Proposition *5-1*

The 1-level scheme is stable if and only if $|\lambda_{L1}|<1$ *this is equivalent to* $\sigma\in(0,1/2)$.

Remark *5-1*

Similar results are obtained for 2 and 3-level smoothing in the section 6.

We used the similarity between finite difference and finite element for uniform discretisation to use the operator L for our test problem.

VI. Application of the Hiearchical Smoothing Residue Method to Elliptic Problems

Abstract - The aim of this part is to confirm the efficiency of the hierarchical smoothing method for some model problems. The analysis of the method is made. Numerical methods are carried out and compared to other methods.

6-1 *Introduction*

Consider the case of elliptic symmetric problems in 2-dimensional space with constants coefficients. The operator associated is symmetric positive definite and could be tackle in a similar way to that presented here. The stationary problems are solved by the stationnarisation procedure. We consider the model problem the Poisson equation in the unit square Ω.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)\ in\ \Omega \qquad (6\text{-}1)$$

$$u \quad I\partial\Omega=\psi(s) \qquad (6\text{-}2)$$

The corresponding stationnarisation scheme reads:

$$\frac{u_{ij}^{p+1}-u_{ij}^{p}}{\tau}=-f_{ij}+\frac{L_h(u_h^p)}{h^2}, p=1,2\ldots\ldots \qquad (6\text{-}3)$$

$$u_{ij}^0=\varepsilon_{ij}^0 \qquad (6\text{-}4)$$

$$u_{ij}^{p+1}=\psi_{ij}\ \ on\ \Gamma \qquad (6\text{-}5)$$

Where $h=\frac{1}{M}$ the uniform space step x_i = ih, y_j = jh

$\frac{L_h(u_h)}{h^2}$ is the finite difference or finite element operator of order 2 associated.

To the elliptic operator with $\|L_h\|=O(1)$. The corresponding stationary scheme is $(6-6)\ L_h(u_h)=h^2 f_h$. The initial condition (6-3) is arbitrary and can be taken to be zero u_{ij}^p is an approximation of u_{ij} *the error* $\varepsilon_{ij}^p=u_{ij}-u_{ij}^p$.

From (6-3) and (6-5) we have the following relations for the error.

$$\frac{\varepsilon_{ij}^{p+1}-\varepsilon_{ij}^{p}}{\tau}=\frac{L_h(\varepsilon_{ij}^p)}{h^2}, p=1,2\ldots\ldots \qquad (6\text{-}7)$$

$$\varepsilon_{ij}^{p+1}=0\ on\ \Gamma \qquad (6\text{-}8)$$

$$\varepsilon_{ij}^0=u_{ij}-u_{ij}^0 \qquad (6\text{-}9)$$

It is well known that the sequence (u^p) converges if and only if the error sequence converges to 0. We are going to carry out a Fourier analysis which respect to an orthonormal basis of Eigen vectors of the Operator L_h .
Let ψ_{ij} be an eigenvector of L_h associated to λ_{ij}
Then $\alpha_{ij}=1+\lambda_{ij}$
Is an eigenvalue of $I+\sigma L_h$ associated to the same eigenvectors.
Where $\sigma=\frac{\tau}{h^2}$

We suppose there exists an orthonormal basis $\psi^{m,n}$, of the operator L_h

We have (6.10) $\varepsilon_0 = \sum_{m,n} c^0_{m,n}\, \psi^{m,n}$

$$(6.11)\ \varepsilon^p = \sum_{m,n} c^0_{m,n}\alpha^p_{m,n}\psi^{m,n}$$

We can deduce the following estimate

$$(6.12)\ \frac{\|\varepsilon^p\|}{\|\varepsilon^0\|} \le \max_{m,n}|\alpha_{m,n}|^p$$

And it follows

Proposition 6-1

The Euler schemes converges if and only if $\max_{m,n}|\alpha_{m,n}|^p < 1$

VII. Analysis of the Euler Schemes RK1

7-1 *Spectral analysis of Finite Elements discretisation and Finite difference of Order 2*

Let us consider the discretisation for our model problem by finite element or difference of order 2 :

$$R_h(u_{m,n}) = u_{m-1,n} + u_{m,n-1} - 4u_{m,n} + u_{m+1,n} + u_{m,n+1}$$

The Euler scheme reads

$$u^{p+1}_{m,n} = u^p_{m,n} + \sigma R_h(u^p_{m,n}) - f_{m,n}$$

Lemma 7-1

The modes $e^{i(\theta_1+\theta_2)}$ are eigenvectors of the operator R_h associated to the eigenvalues

$$-4\left(sin^2\frac{\theta_1}{2} + sin^2\frac{\theta_2}{2}\right), \theta_1, \theta_2 \in (0,\pi)$$

Remark *7-1*

The modes $e^{i(\theta_1+\theta_2)}$ are eigenvectors of the operator $I + \sigma R_h$ associated to the eigenvalues

$$1-4\sigma\left(sin^2\frac{\theta_1}{2} + sin^2\frac{\theta_2}{2}\right), \theta_1, \theta_2 \in (0,\pi)$$

The minimal value is $1 - 8\sigma\ attained\ for\ \theta_1 = \theta_2 = \pi$

It then follows that $\sigma_{max} = \frac{1}{4}$

The problem of the Euler schemes are to find the values σ_{max} and σ_{opt}

For the maximal attenuation of residues that can be expressed as

$$\max_{\theta_1,\theta_2}|\lambda(\sigma_{opt},\theta_1,\theta_2)| = \min_{\sigma\in[0,\sigma max]}\max_{\theta_1,\theta_2}|\lambda(\sigma,\theta_1,\theta_2)| \qquad (7\text{-}2)$$

For the Euler schemes due to the symmetry, the critical frequency is obtained for $\theta_1, = \theta_2 = \pi$

Remark *7-2*

We can prove that for the range $\left[\frac{2\pi}{3},\pi\right]$ $\sigma_{opt} = \frac{1}{7}$

VIII. Analysis of the HSRM 2-level

Consider the operator $L_2 = \sigma SoR \quad L = \sigma R$ and the corresponding schemes\

$$u^{p+1} = u^p - L_2(u^p) \quad (8\text{-}1)$$

The eigenvectors of S and L being the same the amplification factor is $1 - \lambda_S.\lambda_L$

Proposition 8-1

The optimal value of σ, σ_{opt} for the scheme HSRM for the 2- level is $\sigma_{opt} = \frac{4}{7}$

Proof

This follows from the optimisation of the amplification factor in the range $\left[\frac{2\pi}{3}, \pi\right]$

Remark 8-1

We can observe that σ_{opt} *is better for the* $2 -$ *level smoothing*

IX. Analysis of the 3-level HSRM

We proceed as in the case of 2-level.

Let $L_3 = SoL_2$

Proposition 9-1

The 3-level HSRM conserves the breaking values of 1-level and 2-level and the optimal value correspond to $\theta_c = \text{Arccos}\left(\frac{1}{3}\right)$ with $\sigma_{opt} = 1.67357$ for a range containing θ_c .

X. Analysis of the HSRM of type Runge Kutta of Order 4

We consider the stationnarisation schemes of the form $u^{p+1} = u^p - \widetilde{L_i}(u^p), L_i \in \{L_1, L_2, L_3\}\ \widetilde{L_i} = g(L_i)$. Where g is the characteristic function of the Runge Kutta method of order 4. Given by $g(z) = a1a2a3a4.z^4 + a2a3a4z^3 + a3a4z^2 + a4z$.

With

$$(a1, a2, a3, a4) = \left(\frac{1}{10}, \frac{13}{50}, \frac{1}{2}, 1\right)$$

For the corresponding HSRM we have the following amplification factors

$$\lambda(\theta_1, \theta_2, \sigma) = 1 + g\left(-8\sigma\phi_{L_i}(\theta_1, \theta_2)\right)$$

$$\phi_{L_i}(\theta_1, \theta_2) = \begin{cases} \frac{1}{2}\left[\left(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2}\right)\right] \text{for } L_1 \\ \frac{1}{4}\left[\left(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2}\right)\left(\left(\cos^2\frac{\theta_1}{2} + \cos^2\frac{\theta_2}{2}\right)\right)\right] \text{for } L_2 \\ \frac{1}{8}\left[\left(\sin^2\frac{\theta_1}{2} + \sin^2\frac{\theta_2}{2}\right)\left(\left(\cos^2\frac{\theta_1}{2} + \cos^2\frac{\theta_2}{2}\right)^2\right)\right] \text{for } L_3 \end{cases}$$

We can then deduce that

Proposition 10-1

The RK4 version of HSRM schemes conserve the critical frequencies of the RK1 homologue.

Remark 10-1

In a fort coming paper the numerical analysis of Euler Richardson and Douglas-Rachford are done and numerical tests carried out.

XI. Numerical Results

Numerical tests have been done for RK4 for the smoothers N1, N2, N3 associated to $I + L_1$, $I + L_2$, $I + L_3$ where $L_i = g(L_i^0), L_i^0$ being the operators associated to the $RK1$ versions

The following numerical results have been obtained

$$\begin{array}{lll} \quad\quad\quad\; \mathrm{I+L1} & \mathrm{I+L2} & ,\mathrm{I+L3} \\ \sigma max = 0.3485 & 2.7883 & 5.5766 \\ \sigma opt \;= 0.0487 & 0.4101 & 2.1097 \end{array}$$

We note that the HSRM associated to the RK1 converges but for the RK4. Convergence still difficult to observe. In a forthcoming paper more numerical simulations are given and compare to other methods.

We also observe that the combinations of schemes has better convergence rate than that of individual component N1, N2, N3. The rate of convergence of the different combinations is summarized in the table below.

Rate of Convergence for the Schemes HSRM RK1.

T/FR	1	2	3	4
N1	0.7308	0.7313	0.4322	0.7313
N2	0.2390	0.2380	0.1457	0.2380
N3	0.9028	0.9035	0.6031	0.9035
N1oN2	0.1222	0.1238	0.0630	0.1231
N2oN3	0.6176	0.6184	0.2607	0.6184
N1oN2oN3	0.1059	0.1035	0.0380	0.1035

T/FR	5	6	7	8
N1	0.7327	0.7272	0.4322	0.7316
N2	0.2402	0.2369	0.1457	0.2369
N3	0.9042	0.901	0.6031	0.9107
N1oN2	0.1234	01234	0.063	0.1243
N2oN3	0.6245	0.6173	0.2607	0.6200
N1oN2oN3	0.1066	0.1045	0.038	0.1039

XII. Discussion and Results

We can note that the combination of the HSRM have better rate of convergence. The main objective of this method is to use HSRM with low complexity to obtain the efficiency of the multigrid. There is some hope for the RK1 but the version RK4 still to work out.More simulations are to be done and compared to classical method. Deeper analysis and development are done in a fortcoming paper.

ACKNOWLEDGMENTS

We acknowledge the financial support of the ministry of high education through the grant to research and that of the University of Calabar during my visit there.

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