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Several Treasures of the Queen of Sciences

Leszek W. Guła
Lublin-POLAND
Corresponding author email id: yethi@wp.p

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Abstract **– The Fermat's Last Theorem (FLT). The Guła's Theorem. The Goldbach's Theorem. The Conclusions. Supplement — two short proofs: of FLT for n = 4 and of the Diophantine Inequalities.**

Keywords **– Algebra of Sets, Diophantine Equations, Diophantine Inequalities, Exponential Equations, Fermat Equation, Goldbach Conjecture, Greatest Common Divisor, Newton Binomial Formula.**

MSC **— Primary: 11D41, 11P32; Secondary: 11A51, 11D45, 11D61.**

I. INTRODUCTION

The cover of this issue of the Bulletin is the frontispiece to a volume of Samuel de Fermat's 1670 edition of Bachet's Latin translation of Diophantus's Arithmetica. This edition includes the marginalia of the editor's father, Pierre de Fermat. Among these notes one finds the elder Fermat's extraordinary comment in connection with the Pythagorean equation $x^2 + y^2 = z^2$ the marginal comment that hints at the existence of a proof (a demonstratio sane mirabilis) of what has come to be known as Fermat's Last Theorem. Diophantus's work had fired the imagination of the Italian Renaissance mathematician Rafael Bombelli, as it inspired Fermat a century later. [5]

The Guła's Theorem [2] is wider than the Pythagoras's equation and the Diophantus's equation. [3]

The Goldbach's Conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states: Every even integer greater than 2 can be expressed as the sum of two primes. [1]

The short proofs in the Supplement are staggering, that up find difficult believe in them.

II. THE FERMAT'S LAST THEOREM

Theorem 1.

For all $n \in \{3, 4, 5, \dots\}$ and for all $A, B, C \in \{1,2,3, \dots\}$: $A^n + B^n \neq C^n$.

Proof

Suppose that for some $n \in \{3, 4, 5, \dots\}$ and for some $A, B, C \in \{1, 2, 3, \dots\}$:

$A^n + B^n = C^n \Longrightarrow (A + B > C \wedge A^2 + B^2 > C^2$ [4]).

Thus – For some $A, B, C, C - A, C - B, v \in \{1,2,3, \dots\}$:

$$A - (C - B) = B - (C - A) = 2v > 0 \Longrightarrow (C - B + 2v = A \wedge C - A + 2v = B \wedge A + B - 2v = C). \qquad (1)$$

At present we can assume for generality of below that A, B and C are coprime. Then only one number out of a hypothetical solutions $[A, B, C]$ is even. Hence we can assume that $A, C - B \in \{1, 3, 5, \dots\}$.

Let $\{(2a + b)b : a \in [0, 1, 2, \dots] \wedge b \in [3, 5, 7, \dots]\} = \{9, 15, 21, 25, 27, 33, 35, 39, 45, 49, \dots\} \wedge \{3, 5, 7, \dots\} - \{9, 15, 21, 25, 27, 33, 35, 39, 45, 49, \dots\} = \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \dots\} = \mathbb{P}$.

Every even number which is not the power of number 2 has odd prime divisor, hence sufficient that we prove FLT for $n = 4$ and for odd prime numbers $n \in \mathbb{P}$. [6]

A. *Proof for n = 4.*

For all $a, b \in \{0, 1, 2, \dots\}$: the number $\frac{(2a+1)^2 + (2b+1)^2}{2}$ is odd. Thus the number C is odd.

Hence – For some $C, A \in \{1, 3, 5, \dots\}$ and for some $B \in \{4, 6, 8, \dots\}$:

$$(C - A + A)^4 - A^4 = B^4 \Longrightarrow (C - A)^3 + 4(C - A)^2 A + 6(C - A)A^2 + 4A^3 = \frac{B^4}{C - A}.$$

Notice that

$$(C - A)^3 + 4(C - A)^2 A + 6(C - A)A^2 + 4A^3 = \frac{C^4 - A^4}{C - A} = \frac{(C^2 + A^2)(C + A)(C - A)}{C - A}.$$

Thus – For some $k \in \{1, 2, 3, \dots\}$ and for some coprime $e, c, d, h, m \in \{1, 3, 5, \dots\}$:

$$\left[\frac{B^4}{C - A} = \frac{(2^k ecd)^4}{2^{4k-2} d^4} = 4(ec)^4 \wedge h^4 = C - B \wedge 2^k d(2^{3k-2} d^3 + hm) = 2^k ecd = B\right].$$

Therefore – For some relatively prime $e, c \in \{1, 3, 5, \dots\}$ such that $e > c$:

$$4(ec)^4 = (C^2 + A^2)(C + A) \Longrightarrow (C^2 + A^2 = 2e^4 \wedge C + A = 2c^4) \Longrightarrow (C = x + y \wedge A = x - y \wedge C + A = 2x = 2c^4 \wedge x = c^4 \wedge x^2 + y^2 = e^4 \wedge x = c^4 = u^2 - v^2 \wedge y = 2uv \wedge e^2 = u^2 + v^2 \wedge e = p^2 + q^2 \wedge u = p^2 - q^2 \wedge v = 2pq) \Longrightarrow \{x = [(p^2 - q^2)^2 - (2pq)^2] = (c^2)^2 \in \mathbf{0} \wedge y = 4(p^2 - q^2)pq \wedge x^2 + y^2 = [(p^2 - q^2)^2 - (2pq)^2]^2 + 16(p^2 - q^2)^2(pq)^2 = (p^2 + q^2)^4 = e^4 \in \mathbf{1}\} \in \mathbf{0},$$

inasmuch as on the strength of the **Theorem 2** we get

$$(2pq)^2 = (p^2 - q^2)^2 - (c^2)^2 \Longrightarrow p^2 - q^2 = \frac{(2pq)^2 + (2q^2)^2}{2(2q^2)} = p^2 + q^2 \in \mathbf{0}. \spadesuit$$

B. *Proof for* $n \in \mathbb{P}$.

We assume that $4 \nmid B, C$. In view of (1) we will have –

For some $n \in \mathbb{P}$ and for some $C, B, C - A \in \{1, 2, 3, \dots\}$ and for some $C - B, A, v \in \{1, 3, 5, \dots\}$:

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$$\left[(C-B+2v)^n=(C-B+B)^n-B^n \Rightarrow (C-B)^{n-2}v + (n-1)(C-B)^{n-3}v^2+\cdots + 2^{n-2}v^{n-1}+\frac{2^{n-1}v^n}{n(C-B)} = \frac{B}{2}\left[(C-B)^{n-2}+\frac{n-1}{2}(C-B)^{n-3}B + \cdots + B^{n-2}\right] \Rightarrow [n\,|\,v \wedge (n\,|\,B \veebar n\,|\,C)]\right] \wedge$$

$$\left[(C-A+2v)^n=(C-A+A)^n-A^n \Rightarrow (C-A)^{n-2}2v + \frac{n-1}{2}(C-A)^{n-3}(2v)^2+\cdots + (2v)^{n-1}+\frac{(2v)^n}{n(C-A)} = A\left[(C-A)^{n-2}+\frac{n-1}{2}(C-A)^{n-3}A + \cdots + A^{n-2}\right] \Rightarrow [n\,|\,v \wedge (n\,|\,A \veebar n\,|\,C)]\right] \wedge$$

$$\left[(A+B-B)^n+B^n=(A+B-2v)^n \Rightarrow (A+B)^{n-2}(-B) + \frac{n-1}{2}(A+B)^{n-3}(-B)^2+\cdots + (-B)^{n-1} = (A+B)^{n-2}(-2v) + \frac{n-1}{2}(A+B)^{n-3}(-2v)^2+\cdots + (-2v)^{n-1}+\frac{(-2v)^n}{n(A+B)} \Rightarrow [n\,|\,v \wedge (n\,|\,A \veebar n\,|B)]\right].$$

Thus

$$[(n\,|\,B \veebar n\,|\,C) \wedge (n\,|\,A \veebar n\,|\,C) \wedge (n\,|\,A \veebar n\,|\,B)].$$

If $n\,|\,B \equiv 1$, then

$$[(n\,|\,B \veebar n\,|\,C) \equiv 1 \wedge (n\,|\,A \veebar n\,|\,C) \equiv 0 \wedge (n\,|\,B \veebar n\,|\,C) \equiv 1] \in \mathbf{0}.$$

If $n\,|\,C \equiv 1$, then

$$[(n\,|\,B \veebar n\,|\,C) \equiv 1 \wedge (n\,|\,A \veebar n\,|\,C) \equiv 1 \wedge (n\,|\,A \veebar n\,|\,B) \equiv 0] \in \mathbf{0}.$$

If $n\,|\,A \equiv 1$, then

$$[(n\,|\,B \veebar n\,|\,C) \equiv 0 \wedge (n\,|\,A \veebar n\,|\,C) \equiv 1 \wedge (n\,|\,A \veebar n\,|\,B) \equiv 1] \in \mathbf{0}.$$

This is the proof.

III. The Guła's Theorem

Theorem 2.

For each given $g \in \{8,12,16,\dots\}$ or for each given $g \in \{3,5,7,\dots\}$ there exist finitely many pairs (u,v) of positive integers such that:

$g=\left(\frac{g+q^2}{2q}\right)^2-\left(\frac{g-q^2}{2q}\right)^2=(u+v)(u-v)=\frac{g}{q}(u-v)=\frac{g}{q}q=g \Rightarrow g^2=(u^2-v^2)^2=(u^2+v^2)^2-(2uv)^2$, where $q\,|\,g$ and $q<\sqrt{g}$ and - $q,\frac{g}{q}\in\{2,4,6,\dots\}$ with even g or $q\in\{1,3,5,\dots\}$ with odd g. [2]

IV. The Goldbach's Theorem

On the strenght of the proof of the Goldbach's Conjecture [2], [3] and of three theorems **2**, **3** and **4** we get –

Theorem 6.

For all $p,q \in \mathbb{P}$ and for some relatively prime $u,v \in \{1,2,3,\dots\}$ such that $p>q$ and $u-v$ is positive and odd: [8]

$$pq=\left(\frac{p+q}{2}\right)^2-\left(\frac{p-q}{2}\right)^2=u^2-v^2=(u+v)(u-v) \Rightarrow \left[\left(pq,\frac{p^2-q^2}{2},\frac{p^2+q^2}{2}\right) = (u^2-v^2,2uv,u^2+v^2) \wedge p = u+v \wedge q=u-v \wedge 2p = 2u+2v \wedge 2q=2u-2v \wedge p+q = 2u \wedge p-q = 2v \wedge (p+q=2u=8,10,12,\dots) \wedge (p-q=2v=2,4,6,\dots)\right].$$

Proof.

It is easy to verify that

$$4^2-1^2=5\cdot3 \Rightarrow (5+3=8 \wedge 5-3=2),$$
$$5^2-2^2=7\cdot3 \Rightarrow (7+3=10 \wedge 7-3=4),$$
$$6^2-1^2=7\cdot5 \Rightarrow (7+5=12 \wedge 7-5=2),$$
$$7^2-4^2=11\cdot3 \Rightarrow (11+3=14 \wedge 11-3=8),$$
$$8^2-3^2=11\cdot5 \Rightarrow (11+5=16 \wedge 11-5=6),$$
$$8^2-5^2=13\cdot3 \Rightarrow (13+3=16 \wedge 13-3=10),$$
$$9^2-2^2=11\cdot7 \Rightarrow (11+7=18 \wedge 11-7=4),$$
$$9^2-4^2=13\cdot5 \Rightarrow (13+5=18 \wedge 13-5=8),$$
$$10^2-3^2=13\cdot7 \Rightarrow (13+7=20 \wedge 13-7=6),$$
$$10^2-7^2=17\cdot3 \Rightarrow (17+3=20 \wedge 17-3=14),$$
$$11^2-6^2=17\cdot5 \Rightarrow (17+5=22 \wedge 17-5=12),$$
$$11^2-8^2=19\cdot3 \Rightarrow (19+3=22 \wedge 19-3=16),$$
$$12^2-5^2=17\cdot7 \Rightarrow (17+7=24 \wedge 17-7=10),$$
$$12^2-7^2=19\cdot5 \Rightarrow (19+5=24 \wedge 19-5=14),$$
$$13^2-6^2=19\cdot7 \Rightarrow (19+7=26 \wedge 19-7=12),$$
$$13^2-10^2=23\cdot3 \Rightarrow (23+3=26 \wedge 23-3=20),$$
$$14^2-3^2=17\cdot11 \Rightarrow (17+11=28 \wedge 17-11=6),$$
$$14^2-9^2=23\cdot5 \Rightarrow (23+5=28 \wedge 23-5=18),$$
$$15^2-2^2=17\cdot13 \Rightarrow (17+13=30 \wedge 17-13=4),$$
$$15^2-4^2=19\cdot11 \Rightarrow (19+11=30 \wedge 19-11=8),$$
$$15^2-8^2=23\cdot7 \Rightarrow (23+7=30 \wedge 23-7=16),$$
$$16^2-3^2=19\cdot13 \Rightarrow (19+13=32 \wedge 19-13=6),$$
$$17^2-6^2=23\cdot11 \Rightarrow (23+11=34 \wedge 23-11=12),$$
$$17^2-12^2=29\cdot5 \Rightarrow (29+5=34 \wedge 29-5=24),$$
$$17^2-14^2=31\cdot3 \Rightarrow (31+3=34 \wedge 31-3=28),$$
$$18^2-5^2=23\cdot13 \Rightarrow (23+13=36 \wedge 23-13=10),$$
$$18^2-11^2=29\cdot7 \Rightarrow (29+7=36 \wedge 29-7=22),$$
$$18^2-13^2=31\cdot5 \Rightarrow (31+5=36 \wedge 31-5=26),$$

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$$19^2 - 12^2 = 31 \cdot 7 \Longrightarrow (31 + 7 = 38 \wedge 31 - 7 = 24),$$
$$20^2 - 3^2 = 23 \cdot 17 \Longrightarrow (23 + 17 = 40 \wedge 23 - 17 = 6),$$
$$20^2 - 17^2 = 37 \cdot 3 \Longrightarrow (37 + 3 = 40 \wedge 37 - 3 = 34),$$
$$\dots .$$

This is the proof.

V. Conclusions

Theorem 3.

For each pair (u, v) of the relatively prime natural numbers u and v such that $u - v$ is positive and odd there exists exactly one a primitive Pythagorean triple $(u^2 - v^2, 2uv, u^2 + v^2)$ and each the primitive Pythagorean triple arises exactly from one pair (u, v) of the relatively prime natural numbers u and v such that $u - v$ is positive and odd.

Theorem 4.

For each equation $(p,\ q) = (u + v,\ u - v)$ of the relatively prime odd natural numbers p and q such that $p > q$, and of the relatively prime natural numbers u *and* v such that $u - v$ is positive and odd there exists exactly one the primitive Pythagorean triple $\left(pq, \frac{p^2-q^2}{2}, \frac{p^2+q^2}{2}\right) = (u^2 - v^2, 2uv, u^2 + v^2)$ and each this primitive Pythagorean triple arises exactly from one equation $(p, q) = (u + v, u - v)$ of the relatively prime odd natural numbers p and q such that $p > q$, and of the relatively prime natural numbers u and v such that $u - v$ is positive and odd.

Theorem 5.

For all $n \in \{3, 5, 7, \dots\}$ and for all $z \in \{3, 7, 11, \dots\}$ the equation $z^n = u^2 + v^2$ has no primitive solutions $[z, u, v]$ in $\{1,2,3, \dots\}$.

Proof.

Suppose that for some $n \in \{3, 5, 7, \dots\}$ and for some $z \in \{3, 7, 11, \dots\}$ the equation $z^n = u^2 + v^2$ has primitive solutions such that $[z,\ u,\ v] \subset \{1,\ 2,\ 3, \dots\}$. Then the numbers z, u and v are coprime and odd $u - v > 0$.

On the strength of the **Theorem 2** we get –

For some $n \in \{3, 5, 7, \dots\}$ and for some $z \in \{3,7,11, \dots\}$ and for some $d, k \in \{1, 3, 5, 7, 9, \dots\}$ and for some $s, u, v \in \{1, 2, 3, \dots\}$ such that $u - v$ is odd and $k > 2s$:

$$\left[\left(\frac{z^n + d^2}{2d}\right)^2 = \left(\frac{2k + 1 + 4s + 1}{2d}\right)^2 = \left(\frac{k + 2s + 1}{d}\right)^2 = u^2 + \left(\frac{z^n - d^2}{2d}\right)^2 + v^2 \wedge \frac{z^n - d^2}{2d} = \frac{k - 2s}{d}\right] \in \mathbf{0}.$$

because

$$[4 \mid (k + 2s + 1)^2 \wedge 4 \nmid u^2 + (k - 2s)^2 + v^2]. \spadesuit$$

Golden Nyambuya proved reputedly that – For all $n \in \{3, 5, 7, \dots\}$ the equation $z^n = u^2 + v^2$ has no primitive solutions in $\{1, 2, 3, \dots\}$ with $z \in \{3, 5, 7, \dots\} - \{3^2, 5^2, 7^2, \dots\}$. [7]

Corollary 1.

For some $n \in \{3, 5, 7, \dots\}$ and for some $z \in \{5,9,13, \dots\}$ and for some prime natural numbers u, v such that $u - v$ is positive and odd:

$$z^n = u^2 + v^2 \Longrightarrow (u^2 - v^2, 2uv, u^2 + v^2).$$

This is the corollary.

Example 1.

$$5^3 = 11^2 + 2^2 \Longrightarrow (11^2 + 2^2, 44, 11^2 + 2^2).$$

Example 2.

$$17^3 = 52^2 + 47^2 \Longrightarrow (52^2 - 47^2, 4888, 52^2 + 47^2).$$

Example 3.

$$29^3 = 145^2 + 58^2 \Longrightarrow (145^2 - 58^2, 16820, 145^2 + 58^2).$$

These are the conclusions.

VI. Supplement

Suppose that for some $p, q, C \in \{1, 3, 5, \dots\}$ and for some $B \in \{2, 4, 6, \dots\}$ such that the numbers p, q, C and B are coprime and $q < p < C$: $(pq)^4 = C^2 - (B^2)^2$.

We assume that the number C is minimal.

On the strength of the **Theorem 2** we get

$$B^2 = \frac{p^4 - q^4}{2} = \frac{p^2 + q^2}{2}(p^2 - q^2) \Longrightarrow \left(\frac{p^2 + q^2}{2} = w^2 \wedge p^2 - q^2 = r^2\right) \Longrightarrow w^2 = \frac{p^2 + q^2}{2} = \frac{(u^2 + v^2)^2 + (u^2 - v^2)^2}{2} = u^4 + v^4 \Longrightarrow w < C,$$

which is inconsistent with minimal C. □

Let U, u, V and v be four mutually relatively prime natural numbers such that $U - V, u - v$ are positive and odd.

If

$$[U^2 - V^2 = A^2 \wedge 2UV = B^2 \wedge U^2 + V^2 = C^2 \wedge (A^2)^2 + (B^2)^2 = (C^2)^2],$$

then on the strength of the **Theorem 2** we get

$$[V^2 = (2uv)^2 = U^2 - A^2 = C^2 - U^2 \wedge U = u^2 + v^2 \wedge u^2 - v^2 = A] \Longrightarrow$$
$$\left[C = \frac{(2uv)^2 + 2^2}{2 \cdot 2} = (uv)^2 + 1 \wedge u^2 + v^2 = U = \frac{(2uv)^2 - 2^2}{2 \cdot 2} = (uv)^2 - 1\right] \in \mathbf{0}. \ □$$

It's not true in [7] that FLT for $n = 4$ can be written equivalently as: $A^2 = C^4 - B^4$, where $A = 2UV$ or $A = U^2 - V^2$ because Fermat did not proved his own theorem for $n = 4$. [6]

In the first case we will have – If

$$[2UV = A \wedge U^2 - V^2 = B^2 \wedge U^2 + V^2 = C^2 \wedge A^2 + (B^2)^2 = (C^2)^2],$$

then on the strength of the **Theorem 2** we get

$$[V^2 = (2uv)^2 = U^2 - B^2 = C^2 - U^2 \wedge U = u^2 + v^2 \wedge u^2 - v^2 = B] \Longrightarrow$$
$$\left[C = \frac{(2uv)^2 + 2^2}{2 \cdot 2} = (uv)^2 + 1 \wedge u^2 + v^2 = U = \frac{(2uv)^2 - 2^2}{2 \cdot 2} = (uv)^2 - 1\right] \in \mathbf{0}. \ □$$

In the second case we have

$$[U^2 - V^2 = A \wedge 2UV = B^2 \wedge U^2 + V^2$$
$$= C^2 \wedge (U+V)^2(U-V)^2$$
$$= (C^2)^2 - (B^2)^2$$
$$= (C^2 + B^2)(C^2 - B^2) \wedge (U+V)^2$$
$$= C^2 + B^2 \wedge (U-V)^2$$
$$= C^2 - B^2 \wedge U + V$$
$$= u^2 + v^2 \wedge u^2 - v^2 = C \wedge 2uv = B]$$
$$\Longrightarrow$$
$$2UV = (2uv)^2 \Longrightarrow UV = 2u^2v^2$$
$$\Longrightarrow (U = u^2 \wedge V = 2v^2) \Longrightarrow U + V$$
$$= u^2 + 2v^2,$$

which is inconsistent with $U + V = u^2 + v^2$. ▯
This is the supplement.

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Author's Profile

Month and day of birth: 3.14. Defense of the master's thesis (Faculty of Pedagogy and Psychology UMCS) entitled: Efekty Orientacji Zawodowej Uczniow Klas Osmych (Year 1981). Professional title: MSc of techniki, the teacher. From 1981: invalidity pension. E-mail: yethi@wp.pl