International Journal of Innovation in Science and Mathematics
Volume 3, Issue 6, ISSN (Online): 2347–9051

Estimation of Bitopological Index of Trees

Bindhu K. Thomas

***Abstract* – Bitopological index βτ(G) of a finite graph G is the minimum cardinality of a set X such that G is a bitopological graph with respect to X. In this paper we find an optimization of bitopological index of paths and estimate the bitopological index of trees with order upto six.**

***Keywords* – Set-Indexer, Bitopological Graph, Bitopological Index.**

I. Introduction

For all terminology and notation in graph theory, which are not defined in this paper, we refer the reader to F.Harary [1]. Unless mentioned otherwise, all graphs considered in this paper are simple and self-loop free. Given a graph $G = (V, E)$ we can relate it to different topological structures. In 1967, J.W. Evans *et.al* [2] conceived this idea and they proved that there is a one to one correspondence between the set of all topologies on a set X with n points and the set of all transitive digraphs with n points. In 1968, T.N.Bhargav and T.J. Ahlborn[3] analysed the topological spaces associated with digraphs. According to them a subset A of V(D) is open if and only if for every pair of points $i, j \in V$ with j in A and i not in A, (i, j) is not a line in D where D is a digraph. E. Sampathkumar[4][9] extended their notion to the case in which the point set is infinite. In 1983, Acharya [5] established another link between graph theory and point set topology. He proved that for every graph G, there exists a set X and a set-indexer $f: V(G) \to 2^X$ such that the family $f(V)$ is a topology on X. Here we consider a special case of set-indexed graphs in which both $f(V)$ and $f^{\oplus}(E) \cup \{\phi\}$ are topologies on the ground set X. We estimate the minimum cardinality of the ground set X with which trees of order upto six admit a bitopological set-indexer.

II. Preliminaries & Notations

Definition 2.1 [6] Let $G = (V, E)$ be a graph, X be a nonempty set and 2^X denotes set of all subsets of X. A set-indexer of G is an injective set-valued function $f: V(G) \to 2^X$ such that the function $f^{\oplus}: E(G) \to 2^X - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where $\oplus$ denotes the symmetric difference of sets.

Definition 2.2 [10] A graph $G = (V, E)$ is called a bitopological graph if there exists a set X and a set-indexer f on G such that both $f(V)$ and $f^{\oplus}(E) \cup \{\phi\}$ are topologies on X. The corresponding set-indexer is called a bitopological set-indexer of G.

Definition 2.3 [10] Bitopological index $\beta_\tau(G)$ of a finite graph G is the minimum cardinality of the ground set X such that G is bitopological with respect to X.

Proposition 2.4 [10] $\beta_\tau(G) \geq \left\lceil \frac{\log\frac{p+q+1}{2}}{\log 2} \right\rceil$

III. Bitopological Index of Trees

It is a tedious task to find the bitopological index of an arbitrary tree. It may even an NP- complete problem. However we find the bitopological index β_τ of the classes of trees with order upto six and diameter less than or equal to five. We denote a tree of order n and diameter d by T_n^d. Before finding the bitopological index of trees we give an optimization of bitopological index of paths.

Theorem 3.1. For a path P_n, $\left\lceil \frac{\log n}{\log 2} \right\rceil \leq \beta_\tau(P_n) \leq n - 1$.

Let $X = \{1,2,3, \dots n-1\}$
Define $f: V \to 2^X$ by
$f(v_1) = \{\emptyset\}$
$f(v_2) = \{1,2,3, \dots n-1\}$
For $i \geq 2$
$f(v_{2i}) = \{1,2,3, \dots, n+1-i\}$
$f(v_{2i+1}) = \{1,2,3, \dots i\}$
f is a bitopological set-indexer. Hence by the definition of bitopological index
$\beta_\tau(P_n) \leq n - 1$.

Also by Proposition 2.4 $\beta_\tau(P_n) \geq \left\lceil \frac{\log\frac{n+n-1+1}{2}}{\log 2} \right\rceil$

That is , $\beta_\tau(P_n) \geq \left\lceil \frac{\log n}{\log 2} \right\rceil$

Thus we have $\left\lceil \frac{\log n}{\log 2} \right\rceil \leq \beta_\tau(P_n) \leq n - 1$.

Remark 3.3 $\beta_\tau(T_2^1) = \beta_\tau(P_2) = 1$ and $\beta_\tau(T_3^2) = \beta_\tau(P_3) = 2$.

Theorem 3.2 Bitopological Index $\beta_\tau(T_n^d) \leq n - 1$ *for* $n \leq 6$ *and* $d \leq 5$.

Proof: We consider each classes separately as follows.

1 . $\beta_\tau(T_4^2) = 2$
$T_4^2 \equiv K_{1,3}$
Let v_1, v_2, v_3, v_4 are the vertices of $K_{1,3}$ with $\Delta(v_1) = 3$. Define a set-indexer f by
$f(v_1) = \{\emptyset\}$
$f(v_2) = \{x_1\}$
$f(v_3) = \{x_2\}$

$f(v_4) = \{x_1, x_2\}$. Then f is a bitopological set-indexer with the ground set $X = \{x_1, x_2\}$.

Therefore $\beta_\tau(T_4^2) \equiv \beta_\tau(K_{1,3}) = 2$.

2. $\beta_\tau(T_4^3) = 3$

$T_4^3 \equiv P_4$

By Proposition 2.4, $\left\lceil \frac{\log 4}{\log 2} \right\rceil \le \beta_\tau(P_4) \le 4 - 1$

That is $2 \le \beta_\tau(P_4) \le 3$

Let $X = \{x_1, x_2\}$

Then the possible vertex valuation of the four vertices are given below

$\{\emptyset\}$,	$\{x_1\}$,	$\{x_2\}$,	$\{x_1, x_2\}$
$\{x_1\}$,	$\{\emptyset\}$,	$\{x_2\}$,	$\{x_1, x_2\}$
$\{x_1\}$,	$\{x_2\}$,	$\{\emptyset\}$,	$\{x_1, x_2\}$
$\{x_1\}$,	$\{x_2\}$,	$\{x_1, x_2\}$,	$\{\emptyset\}$
$\{x_1\}$,	$\{x_1, x_2\}$,	$\{x_2\}$,	$\{\emptyset\}$

The edge valuations corresponding to this vertex valuations are not injective and hence f is not a bitopological set- indexer.

Therefore $\beta_\tau(P_4) = 3$. That is $\beta_\tau(T_4^3) = 3$.

3. $\beta_\tau(T_5^2) = 3$

$T_5^2 \equiv K_{1,4}$

Let v_1, v_2, v_3, v_4, v_5 are the vertices of $K_{1,4}$ with $\Delta(v_1) = 4$. Then the set-indexer f defined by

$f(v_1) = \{\emptyset\}$

$f(v_2) = \{x_1\}$

$f(v_3) = \{x_2\}$

$f(v_4) = \{x_1, x_2\}$

$f(v_5) = \{x_1, x_2, x_3\}$ is bitopological and hence $\beta_\tau(T_5^2) = \beta_\tau(K_{1,4}) = 3$.

4. $\beta_\tau(T_5^3) = 3$

Let T_5^3 be a tree with three pendant vertices v_1, v_2 *and* v_3. Let v_4 be the vertex with degree 3 . Suppose v_4 is adjacent to v_1, v_2, v_5 and v_3 is adjacent to v_5.

Then the set-indexer f defined by

$f(v_5) = \{\emptyset\}$

$f(v_2) = \{x_1\}$

$f(v_4) = \{x_2\}$

$f(v_1) = \{x_1, x_2\}$

$f(v_3) = \{x_1, x_2, x_3\}$ is bitopological and hence $\beta_\tau(T_5^3) = 3$.

5. $\beta_\tau(T_5^4) = 4$

Now $T_5^4 \equiv P_5$

Let $X = \{x_1, x_2, x_3\}$

Non-isomorphic topologies on $X = \{x_1, x_2, x_3\}$ consisting of five open sets are

$T_1 = \{\{\emptyset\}, \{x_1\}, \{x_2\}, \{x_1, x_2\}, X\}$

$T_2 = \{\{\emptyset\}, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}, X\}$

Then the possible vertex valuation of the five vertices corresponding to T_1 are given below[7][8]

$\{\emptyset\}, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}$

$\{x_1\}, \{\emptyset\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}$

$\{x_1\}, \{x_2\}, \{\emptyset\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}$

$\{x_1\}, \{x_2\}, \{x_1, x_2\}, \{\emptyset\}, \{x_1, x_2, x_3\}$

$\{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{\emptyset\}$

$\{x_1\}, \{\emptyset\}, \{x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2\}$

$\{x_2\}, \{\emptyset\}, \{x_1, x_2, x_3\}, \{x_1, x_2\}, \{x_1\}$

$\{x_1\}, \{x_1, x_2, x_3\}, \{x_2\}, \{x_1, x_2\}, \{\emptyset\}$

$\{x_1, x_2, x_3\}, \{x_1\}, \{x_1, x_2\}, \{x_2\}, \{\emptyset\}$

The induced edge valuations corresponding to this assignment of sets do not form a topology on the edge set.. Similarly the possible vertex valuation of the five vertices corresponding to T_2 also does not form a topology on its edges. Therefore $\beta_\tau(P_5) = 4$ since $3 \le \beta_\tau(P_5) \le 4$. Thus $\beta_\tau(T_5^4) = 4$.

6. $\beta_\tau(T_6^2) = 3$

$T_6^2 \equiv K_{1,5}$

Let v_1, v_2, v_3, v_4, v_5 *and* v_6 are the vertices of $K_{1,5}$ where v_1 is the central vertex with degree five. Take $X = \{x_1, x_2, x_3\}$. Consider any topology on X with six elements. Assign empty set to v_1 and then arbitrarily assign other open sets to remaining five vertices of $K_{1,5}$. Then it is a bitopological assignment and hence $\beta_\tau(T_6^2) = 3$.

7. $\beta_\tau(T_6^3) = 3$

T_6^3 is a tree with four pendant vertices. Then either it has a vertex of degree four or it has two vertices of degree three. In both the cases it contains P_4 as an induced subgraph and hence $\beta_\tau(T_6^3) > 2$.

Case I

Let $d(v_3) = 4$ *and* v_1, v_2, v_3, v_4 are the vertices of the induced path P_4 . Also let v_5 *and* v_6 are the two pendant vertices adjacent to v_3. Take $X = \{x_1, x_2, x_3\}$. Then the set-indexer defined by

$f(v_1) = \{x_2, x_3\}$

$f(v_2) = \{x_1, x_2\}$

$f(v_3) = \{\emptyset\}$

$f(v_4) = \{x_1, x_2, x_3\}$

$f(v_5) = \{x_1\}$

$\boldsymbol{f(v_6)} = \{x_2\}$ is bitopological and therefore $\beta_\tau(T_6^3) = 3$.

Case II

Let v_1 *and* v_2 are the two adjacent vertices with degree three. Suppose v_3 *and* v_4 are two pendant vertices adjacent to v_1 and v_5 *and* v_6 are other two pendant vertices adjacent to v_2.

Let $X = \{x_1, x_2, x_3\}$.

The set-indexer defined by

$f(v_1) = \{x_1, x_2, x_3\}$
$f(v_2) = \{\emptyset\}$
$f(v_3) = \{x_1, x_3\}$
$f(v_4) = \{x_2\}$
$f(v_5) = \{x_1\}$
$f(v_6) = \{x_1, x_2\}$

is a bitopological set-indexer. Hence in this case also $\beta_\tau(T_6^3)$ =3.

8. $\beta_\tau(T_6^4)$ = 3 or 4

Case I

If T_6^4 contains P_5 as an induced subgraph and the middle vertex has degree three then assign $\{x_1, x_2, x_3\}$, $\{x_2\}$, $\{\emptyset\}$, $\{x_1\}$, $\{x_2, x_3\}$ to the vertices of the induced subgraph P_5 and assign $\{x_1, x_2\}$ to the pendant vertex which is adjacent to the middle vertex. Then this assignment is bitopological and hence $\beta_\tau(T_6^4)$ =3

Case II

Suppose T_6^4 contains P_5 as an induced subgraph and the first internal vertex has degree three. Then any set-indexer with ground set of cardinality three is not bitopological. Assign $\{x_1, x_3\}$, $\{x_1, x_2, x_3\}$, $\{x_1\}$, $\{x_1, x_2, x_3, x_4\}$, $\{\emptyset\}$ to the vertices of the induced subgraph P_5 and assign $\{x_1, x_2\}$ to the pendant vertex which is adjacent to the vertex assigned to $\{x_1, x_2, x_3\}$. Then this assignment is bitopological and hence $\beta_\tau(T_6^4)$ =4.

9. $\beta_\tau(T_6^5)$ =5

$T_6^5 \equiv P_6$. Let $X = \{x_1, x_2, x_3, x_4\}$. If $f(V)$ is a topology on X with six open sets then the edge valuation corresponding to the set-indexer f does not form a topology on the edge set. Therefore $\beta_\tau(T_6) = 5$.

IV. Conclusion

It would be worthwhile to pursue finding newer classes of bitopological graphs. Characterization of bitoplogical trees and calculation of bitopological index of arbitrary trees are further areas of research. Embedding of an arbitrary tree in a bitopological tree and the calculation of chromatic number, clique number and independent number is another area of investigation.

References

[1] F.Harrary(1969), *Graph Theory*, Addison Wesley.

[2] J.W.Evans, F. Harrary and M.S. Lynn. *On Computer enumeration of finite topologies*. Assoc. Comp. Mach. 10(1967), 295-298.

[3] T.N. Bhargav and T.J.Ahjborn. *On topological spaces Associated with digraphs*.Acta, Math. Acad. Sc. Hungar. 19(1968), 47-52

[4] E. Sampathkumar, *Transitive Digraphs and Topologies on a set*, Journal of the Karnatak University- Science (1972), 266-273.

[5] B.D.Acharya, (1983), *Set-valuation of Graphs and Their Applications*, MRI Lecture Notes in Applied Mathematics, No.2

[6] B.D.Acharya,(2001), *Set-Indexers of a graph and Graceful Graphs*, Bulletin, .Allahabad Math.Soc., 16,1- 23.

[7] H.Sharp(1968), *Cardinality of Finite Topologies*, J. Combinatorial Theory,5,82- 86.

[8] D.Stephen(1968), *Topology on Finite Sets*, Amer.Math. Monthly, 75, 739-741.

[9] E.Sampathkumar and K.H.Kulkarni, *Transitive digraphs and topologies on a se*t, Journal of the Karnatak University- Science, 18 (1973), 266- 273.

[10] Bindhu K Thomas, *Advanced Studies on Labelling of Graphs and Hypergraphs* and Related topics, Ph.D Thesis(2009).

Author's Profile

Dr. Bindhu K. Thomas
Assistant Professor; PG and Research Department of Mathematics, Mary Matha Arts and Science College, Mananthavady, Kerala-India
E-mail- bindhukthomas@gmail.com

Completed M.Sc in Mathematics from Devagiri College, Calicut University in 2011, B. Ed from Kannur University in 2012 and M.Phil in 2003 from Calicut University. Awarded Ph.D in 2010 on the topic "Advanced Studies on Set-Labeling of Graphs and Hypergraphs and Related Topics" as a research student of Mathematics Research Centre of Mary Matha Arts and Science College, Mananthavady under kannur University.