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Existence and Regularity Properties of a Class of Elliptic Boundary Value Problem

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***Abstract* – We prove in this paper the existence and unicity of the solutions of a class of nonlinear elliptic problem using monotone accretive operators.**

***Keywords* – Elliptic Problem, Existence Unicity, Monotone Operator.**

I. Introduction

We deal in this paper with a class of elliptic problems with nonlinear Newton boundary condition. The existence and regularity of the solutions are proved using, The theory of monotone operators in reflective Banach spaces into their duals.

In [8] an analysis of the finite element approximation of the solution of the problem is carried out.

The effect of polygonal approximation of a smooth domain on finite element accuracy is studied in [9].

Numerical schemes are proposed in approximating the nonlinear problem are proposed, the convergence analysis is carried out in [10]

II. Preliminaries

The Newton boundary type modeling the effects of disturbances near the boundary (Boundary layer, turbulence).

In general the form of the exact effect is not known. Many approximation procedures can be used (Taylor approximations asymptotic expansions not necessary polynomials).

Feistauer considered in [1,2], the polynomials) growth which is really natural and yield even good results, we consider a large class of functions including the one of [1,2]. Our considerations are justified by the fact that the disturbances

i) Could be of global nature

ii) The exact nature of the functions describing the turbulence near the

boundary could be better approximated by some asymptotic developments where the small terms are nonpolynomials of the form $|x|^{\beta} In|x|, |x|^{\beta} \cdot (\sin x)^{\alpha}, |x|^{\beta} \cdot (\cos x)^{\alpha}$ and some other asymptotic forms. Our class of *functions is described by some* assumptions.

Following the work of Feistauer [1,2], we proved the existence, regularity and unicity of the solutions of the stated problems.

III. Formulation of the Problem

3.1 Setting of the problem

We consider a bounded smooth domain Ω of $R^N (N = 2.3)$.

Let $H_o^{m \cdot p}(\Omega), H_o^{m \cdot p}(\Omega)$ denote classical Sobolev spaces. $m \in \mathrm{N}, p \in]1, +\infty[$

The spaces are simply denoted H^m and H_o^m. Let *f* be in the dual space of H_o^m and consider the problem we consider the problem say (P)

Find u in $H^1(\Omega)$ satisfying the Poisson equations.

$$(3.1.1) - \Lambda u = f \quad in\ \Omega$$

$$(3.1.2) \frac{\partial u}{\partial n} + \beta(u)u\Big|_{\Gamma} = g$$

$$g \in L^2(\Gamma) \cdot$$

3.2 Variation formulation

The problem (P) is equivalent to:

Find u in $H^1(\Omega)$

Such that:

for all in $H^1(\Omega)$. We can then define the following nonlinear operator from $H^1(\Omega)$ into its dual by:

$$(3.2.2)\ (Au, v) = \int_{\Omega} \nabla u \nabla v dx + \int_{r} \beta(u) u v ds$$ for all v in $H^1(\Omega)$ $Au \in [H(\Omega)]$.

3.3 Monotone operators of Banach spaces

Let X be a Banach space supposed reflexive.

Let X′ be the dual space of X equipped respectively with the norms $\|\cdot\|_X, \|\cdot\|_{X'}$

3.3.1 Definition

An operator A defined from a Banach space X to its dual X′ is monotone if for all u and v in X, the quantity $(Au = Av, u - v) = 0$ implies $u = v$. The operator is said hemi-continuous at u:

If the application from

$[0,1] \rightarrow H^1(\Omega)$

$t \rightarrow A((1-t)u + tv)$ is weakly continuous. For all v in $H^1(\Omega)$.

3.3.2 Definition

The operator A is said coercive if there exists a positive function q defined on R^+, such that Q(t) tends to infinity with t

$(Au,u)\geq\|u\|_X\, q(\|u\|_X)$ for all u in $H^1(\Omega)$. We state in the following lemmata necessary for the proof of our existence theorem.

3.4 Lemma

If A maximal monotone and coercive operator from a reflective Banach space V into its dual V′, then A is subjective from V and V′.

3.5 Lemma

Any monotone and hemi-continuous, the A is hemi-continuous operator from a B-reflexive space is maximal monotone.

Remark

The proof of these lemmas are found in [9].

3.6 Remark

If the operator A defined from a reflexive B-space into its dual is simply weakly continuous, then A is hemi-continuous.

IV. Existence and Regularity of the Solution of Problem (P)

4.1 Assumptions on data

Assumption H1

We assume that the function β occurring in the boundary condition, (3-1-1) satisfies:

$$\beta(0)=0$$

Assumption H2

For all positive real number r there exist a constant λ such that:

For all u and v in $H^1(\Omega)$

$\|u\|\leq r, \|v\|\leq r,$

$|[\beta(u)u-\beta(v)v]wds|\leq K(r)\|u-v\|1-\|w\|1$

k(r) is a positive constant for all w in $H^1(\Omega)$.

Assumption H3

Let g: $R\rightarrow R$

$t\rightarrow t\beta(r)$

we assume that there exists positive constant α and c such that:

for all $h\succ 0,\ g(t+h)-gt)\geq ch^{\alpha}$ for all t in R.

4.1 Consequence

(a) Under the assumption H1, the function g defined in assumption H3 is an increasing function.

(b) The operator

$L: H^1(\Omega)\rightarrow [H^1(\Omega)]$

$u\rightarrow Lu$

where Lu is defined by:

$(Lu,v)=\int_r [\beta(u)uvds$, for all v in $H^1(\Omega)$.

The operator L is hemi-continuous. From $H^1(\Omega)$, into its dual.

4.2 Existence theorem

Under the assumption H1, the operator A defined in (3.2.2) is maximal monotone.

Proof:

$(Au-Av,u-v)=\int|\nabla(u-v)|^2dx+\int_r(\beta(u)\beta(v)v)(u-v)ds$

for all u and v in $H^1(\Omega)$.

Setting $q=u(u-v)sig(u-v)\ if\ u\neq v$

And $t=v,\ if\ u\succ v, t=u\ if\ u\prec v$ then.

$(\beta(u)u-\beta(v)v)(u-v)=(t+q)-t\beta(t)\geq 0$. Since q ≥ 0 for all u and v.

Since g is increasing by assumption H3 we then deduce from lemma 3.1 that A is monotone..

From assumption H1 and H3 we then deduce that A is hemi-continuous. If then follows from lemma 3.2 that A is maximal monotone. Moreover if $(Au-Av,u-v)=0$ implies by assumption H3 that u = v.

4..2.2 Lemma

The operator A is coercive under the assumption H1 and H3.

Proof:

We have by definition

$$(Au,u)=\int_\Omega|\nabla u|^2dx+\int_r\beta(u)uds\geq\int_\Omega|\nabla u|^2dx+c\int_r|u|^2ds$$

The results follows by the Friedrich inequality. Valid at least for bounded Lipchitz domains. We have for any u in $H^1(\Omega)$.

$$\|u\|_{H^1}\leq c\left[\int_\Omega|\nabla u|^2dx+\int_r|u|^2ds\right]^{1/2}$$

using the imbedding

$L^q(\Omega)\rightarrow L^p(\Omega)$

valid at least for bounded domain for $q\geq p$. It follows easily that:

$(Au,u)/\|u\|_1\geq c\|u\|_1$ at least for $\int_r|u|^2ds\geq 1$

For the case $\int_r|u|^2ds\geq 1$, consider the function

$h(t)=\dfrac{a+t^q}{(a+t)^q}$, for $0\leq t\leq 1, q=\alpha+2, \alpha\geq 0$

The variation of h give $h(t)\geq 1/(1+a)^{q+1}$, taking $a=\int_\Omega|\nabla u|^2dx, t=\int_r|u|^2ds$ we obtained

$$(Au,u)\leq c\left[\int_\Omega|\nabla u|^2dx+\int_r|u|^2ds\right]\Big/(a+1)^{q+1}$$

From the Friedrich inequality, it follows in both cases that:

$\dfrac{(Au,u)}{|u|_1}\rightarrow+\infty$, for $\|u\|_1\rightarrow+\infty$

(i) $(Au-Av,u-v)\geq\int_\Omega|\nabla(u-v)|^2dx+c\int_r|u-v|^{\alpha+2}dx$ for all u and v in $H^1(\Omega)$ c and α are constants given by (1.5).

(ii) $(Au-Av,u-v)\geq\rho(\|u-v\|_1)$

ρ is defined by

where $\rho(t)=\begin{cases} c_1 t^{2+\alpha} si\ 0\le t\le 1 \\ c_1 t^2 si\ t\ge 1 \end{cases}$

Proof:

The results follow immediately from lemma 4.2.2 and assumption H3.

4.2.4 Theorem

Under the assumption H1, H2, H3 the problem (P) has a unique solution u in $H^1(\Omega)$. Furthermore if $f\in W^{k,q}(\Omega), q>2, \Omega$ of class c^k the unique solution u of problem (P) belongs to $W^{k+2,q}(\Omega)$.

Proof:

By the lemma 4.2.1 and 4.2.2 the operator A defined by (3-2-2) is maximal monotone and coercive hence by lemma 3.1 (P) has a unique solution u in $H^1(\Omega)$. Setting $\phi = u|_\Gamma \in H^{1/2}(\Gamma)$.

Rewriting the (2.2.1) in the form:

$$-\Delta u + u = u + f \ in\ \Omega$$

$$u|_\Gamma = \phi$$

It follows by the classical regularity theorem of the Dirichlet problem associated to Δ that u is in $H^2(\Omega)$. Higher order regularity are obtained in the similar way for f and Ω regular.

4.2.5 Corollary

If Ω is a convex polygon with no reentrant angle, then the problem (P) is well posed. Has the following regularity properties: There exists a real $P_o \ge 2$, such that u belongs to $W^{2,p}(\Omega)$ for any $1 < p \le P_o$.

Proof:

The existence of u follows from the result of Grisvard [3].

V. Conclusion

We have extended the result of Feistauer [1] to a larger class of functions defining the boundary conditions (boundary layer turbulence). Under the assumption H1, H2 and H3.

We can obtain the error estimates in Feistauer [1, 2]. It is also easy to prove that these estimates remained true for the solution of out class of data β. A different approach has been used to establish the coercicity of the operator A. The regularity of the solution is also proved.

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