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Optimal Probability of Survival of an Insurer and a Reinsurer under Proportional Reinsurance and Power Utility Preference

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***Abstract* – This study assumed the risk reserve of an insurer and a reinsurer to follow Brownian motion with drift and tackled their optimal probability of survival problem under proportional reinsurance and power utility preference. The insurer's and reinsurer's surplus processes are approximated by Brownian motion with drift and the insurer can purchase proportional reinsurance from a reinsurer. In addition, the insurer and reinsurer are allowed to invest in one risky and one risk-free, assets. We obtained by solving the corresponding Hamilton-Jacobi-Bellman (HJB) equations, the optimized values of the insurer and the reinsurer, optimal investment in the risky asset by both of them and then solved for the discount value, ϕ, that would warrant reinsurance, according to the optimal reinsurance proportion chosen by the insurer.**

***Keywords* – Hamilton-Jacobi-Bellman (HJB) Equation, Insurer, Optimal Investment, Power Utility Function, Probability Of Survival, Proportional Reinsurance, Reinsurer.**

I. Introduction

As investment income has gradually become an important way to increase the profit of insurance company, optimization problems taking both reinsurance and investment into account with different objectives have inspired literally hundreds of researches. Browne [1] considered a diffusion risk model and obtained the optimal investment strategies of exponential utility maximization and ruin probability minimization. Hipp and Plum [2] used the compound Poisson risk model and studied an insurer's optimal policy to minimize the ruin probability. Schmidli [3] used a Brownian motion with drift to model the claim process and obtained the optimal quota-share reinsurance strategy for an insurer. Later, Yang and Zhang [4] considered the optimal investment problem for an insurer with jump-diffusion risk process. Promislow and Young [5] discussed the problem of minimizing the ruin probability subject to both investment and proportional reinsurance strategies for diffusion risk model. Luo et al. [6] studied a similar optimal reinsurance and investment problem in the case of neither short selling nor borrowing. Bai and Guo [7] investigated the optimal proportional reinsurance and investment problem with multiple risky assets.

Cao and Wan [8] considered the optimal reinsurance-investment problem of utility maximization and obtained the explicit solutions for exponential and power utility functions. Gu et al. [9] used the constant elasticity of variance (CEV) model to study the optimal reinsurance and investment problem for an insurer. Zhao et al. [10] discussed the robust portfolio selection problem for an insurer with exponential utility preference. Liang et al. [11, 12] derived the optimal proportional reinsurance and investment strategies for exponential utility maximization under different financial markets. Lin and Li [13] focused on an optimal reinsurance-investment problem for an insurer with jump-diffusion risk model when the stock's price was governed by a CEV model. In Gu et al. [14], the insurer was allowed to purchase excess-of-loss reinsurance and invested in a financial market, and optimal strategies were obtained explicitly. Li et al. [15] investigated the optimal time consistent reinsurance and investment problem under the mean-variance criterion for an insurer. Li and Li [16] took the state dependent risk aversion into account based on Li et al. [15] with the price process of the risky assets satisfying geometric Brownian motion. Zhao et al. [17] considered the optimal excess- of-loss reinsurance and investment problem for an insurer with jump-diffusion risk process under the Heston model. Liang and Bayraktar [18] discussed an optimal reinsurance and investment problem in an unobservable Markov-modulated compound Poisson risk model, where the intensity and jump size distribution were not known but had to be inferred from the observations of claim arrivals. Besides, there are some other interesting topics about the insurance, (Zhang and Ma [19], Xu and Ma [20], Wang et al. [21]).

However, most of the above researches only consider the investment problem for the insurer and ignore the management of the reinsurer. But the reinsurer also faces ruin and needs to invest in a financial market to manage his/her wealth. Thus we study the investment problem for both the insurer and the reinsurer.

From the above literatures, we find that maximizing the expected exponential utility and minimizing the ruin probability are two common investment objectives for the insurer. Moreover, Browne [1], Bai and Guo [7] have shown that maximizing the exponential utility and minimizing the ruin probability produce the same type of investment strategy for zero interest rate. Thus we consider these two objective functions for the reinsurer and examine the equality of the reinsurer's strategy under the two cases.

In this paper, we focus on the optimal probability of survival problem for both the insurer and the reinsurer when the insurer can purchase proportional reinsurance. In our model, the basic claim process is assumed to follow a

Brownian motion with drift. The insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. Furthermore, we consider the correlation between the claim process and the risky asset's price. We first derive the insurer's reinsurance investment strategy for maximizing the probability of survival.

Then with the optimal reinsurance proportion, we consider two optimization problems for the reinsurer: the problem of maximizing the expected power utility of terminal wealth and the problem of maximizing the probability of survival. By solving the corresponding Hamilton-Jacobi-Bellman (HJB) equations, we obtain the explicit solutions and value functions for the optimization problems of the insurer and the reinsurer. Moreover, we illustrate the equality of the reinsurer's optimal investment strategies for the two objective functions..

This paper proceeds as follows; In Section 2, we present formulation of the model. Section 3 gives we derive the optimal probability of survival strategies and value functions for the insurer and the reinsurer's optimization problems. Section 4 concludes the paper.

II. Model Formulation and the Model

Suppose the claim process $C(t)$ of an insurance company is described by;

$$dC(t) = adt - bdZ^{(1)}(t), \quad (1)$$

where a and b are positive constant and $Z^{(1)}(t)$ a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$.Assuming also that the premium rate is;

$$c = (1+\theta)a \quad (2)$$

with safety loading (security risk premium) $\theta > 0$.

Using equation (1), the surplus process of the insurer is given by;

$$dR(t) = \rho dt - dC(t)$$
$$= a\theta dt + bdZ^{(1)}(t). \quad (3)$$

The insurance company has the permission to purchase proportional reinsurance to reduce her risk and pays reinsurance premium continuously at the rate of $(1 + \eta ap(t)$ where $\eta > \theta > 0$ is safety loading of the reinsurer and $r(t)$ is the proportion reinsured at time t.

The surplus of the insurance company is then given as;

$$dR_I(t) = \big(\theta - \eta p(t)\big)adt + b(1 - p(t)d\, Z^{(1)}(t)), \quad (4)$$

for the insurer, and

$$dR_R(t) = \eta p(t)adt + bp(t)d\, Z^{(1)}(t)), \quad (5)$$

for the reinsurer, (Danping et al,[22]).

Assuming that the insurer and reinsurer invest their surplus in a market consisting of two assets: a risky asset (stock) and a riskless asset (bond) which rate of return is a linear function of time, let the prices the riskless and be risky assets $P_0(t)$ and $P(t)$ respectively, then, the equations governing the dynamics of the dynamics of the riskless asset and the risky asset are given by stochastic differential equations;

$$dP_0(t) = (\alpha + \beta t)P_0(t);\ \ P_0(0) = 1, \quad \gamma > 0,\ \ 0 \leq \beta \leq 1, \quad (6)$$

and

$$dP(t) = P(t)\,[\mu dt + \beta dZ^{(2)}(t)], \quad (7)$$

(Osu and Ihedioha, [23]; [24]), respectively.

μ and β denote the appreciation rate (mean) and the volatility of the risky asset, respectively. $Z^{(2)}(t)$ is another standard Brownian motion defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ and,

$$Cov\left(Z^{(1)}(t), Z^{(2)}(t)\right) = \rho t. \quad (8)$$

Both the insurer and the reinsurer hold the risky asset as long as.

$$\mu > (\alpha + \beta t). \quad (9)$$

Let $\pi_I(t)$ represent the amount invested in the risky asset at time t by the insurer and $\pi_R(t)$ the amount invested in the risky asset at time t by the reinsurer. For the insurer, the reinsurance-investment strategy $(p(t), \pi_I(t))$ is called admissible if it is $\mathcal{F}_t$-progressively measurable and satisfies,$0 \leq r(t) \leq 1$, that is;

$$E[\int_0^T \pi_I(t)^2 \mathrm{dt}] < \infty, \quad (10)$$

and for the reinsurer the strategy $(p(t), \pi_R(t))$ is called admissible if it is $\mathcal{F}_t$-progressively measurable and satisfies,$0 \leq p(t) \leq 1$, that is;

$$E[\int_0^T \pi_R(t)^2 \mathrm{dt}] < \infty \quad (11)$$

Assume that $W_I(t)$ and $W_R(t)$ are the total wealth of insurer and the reinsurer, respectively, then their investments in the riskless asset are $(W_I(t) - \pi_I(t))$ and $W_R(t) - \pi_R(t))$, respectively.

For the corresponding admissible strategies, $(p(t), \pi_I(t))$ and $(p(t), \pi_R(t))$, and the policy π, the wealth processes of the insurer and the reinsurer evolve according to the stochastic differential equations (SDEs);

$$dW_I^\pi(t) = \pi_I(t)\frac{dP(t)}{P(t)} + \big(W_I(t) - \pi_I(t)\big)\frac{dP_0(t)}{P_0(t)} + dR_I(t) \quad (12)$$

for the insurer, and

$$dW_R^\pi(t) = \pi_R(t)\frac{dP(t)}{P(t)} + (W_R(t) - \pi_R(t)\,)\frac{dP_0(t)}{P_0(t)} + dR_R(t) \quad (13)$$

for the reinsurer, (Wokiyi, [25]).

Substituting the expressions for, $\frac{dP(t)}{P(t)}$, $\frac{dP_0(t)}{P_0(t)}$, $dR_I(t)$, and $dR_R(t)$, in equations (12) and (13) we get;

$$dW_I^\pi(t) = \pi_I(t)\left[\mu dt + \beta dZ^{(2)}(t)\right] + \big(W_I(t) - \pi It\alpha + \beta t\theta - \eta ptadt + b(1 - p(t)d\,Z1(t)), \quad (14)$$

for the insurer and;

$$W_R^\pi(t) = \pi_R(t)\left[\mu dt + \beta dZ^{(2)}(t)\right] + \big(W_R(t) - \pi Rt\alpha + \beta t + \eta ptadt + bp(t)d\,Z1(t)), \quad (15)$$

for the insurer.

The quadratic variations of the wealth processes of the insurer and the reinsurer are;

$$\langle dW_I^\pi(t)\rangle = \left[\pi_I{}^2(t)\beta^2 + b^2\big(1 - p^2(t)\big) + 2\beta b(1 - pt)\pi It\rho dt \quad (16)$$

$$\langle dW_R^\pi(t)\rangle = [\pi_R{}^2(t)\beta^2 + b^2p^2(t) + \ 2\beta\rho bp(t))\pi_R(t)]dt \quad (17)$$

Suppose the investor has a power utility function, the Arrow-Pratt measure of relative risk aversion (RRA) or coefficient of relative risk aversion is defined as;

$$R(w) = \frac{-wU''(w)}{U'(w)}, \quad (18)$$

where w is the wealth level of an investor. The special case being considered is where the utility function is of the form,

$$U(w) = \frac{w^{1-\phi}}{1-\phi}, 1 \neq \phi \quad (19)$$

which has a constant relative risk aversion parameter ϕ,the investors' (the insurer and the reinsurer) problem can therefore be written as:

$$\left.\begin{aligned} Max_{\pi}\{\mathcal{A}^{\pi}V(t,w)\} &= 0 \\ V(T,w) &= U(w) \end{aligned}\right\} \quad (20a)$$

where

$$V(t,w) = Max_{\pi}E^{(t,w)}[U(W_t^{\pi})] \quad (20b)$$

and $\mathcal{A}^{\pi}$ a generator which in our case shall be derived via Ito lemma,

subject to:

$$dW_I^{\pi}(t) = \pi_I(t)\left[\mu dt + \beta dZ^{(2)}(t)\right] + \left(W_I(t) - \pi_I(t)\right)(\alpha + \beta t) + \left(\theta - \eta p(t)\right)a dt + b(1-p(t)d\,Z^{(1)}(t)),$$

for the insurer and;

$$dW_R^{\pi}(t) = \pi_R(t)\left[\mu dt + \beta dZ^{(2)}(t)\right] + \left(W_R(t) - \pi R t \alpha + \beta t + \eta p t a d t + b p(t) d\,Z1(t)\right),$$

for the reinsurer.

III. The Optimization

The case of the Insurer:

The theorem that follows gives the optimization of the insurer's wealth;

Theorem 1: The optimal policy that maximizes the expected power utility at terminal time T is to invest at each time $t \leq T$;

$$\pi^*(t) = \frac{[\mu-\alpha-\beta t]w_I}{\beta^2\phi} - \frac{b\rho(1-p(t))}{\beta} \quad (21)$$

with optimal proportion reinsure

$$p^*(t) = 1 - \left[\frac{b(b+\beta\rho\pi_I(t))\phi}{\beta}\right]; 0 < \frac{b(b+\beta\rho\pi_I(t))\phi}{\beta} < 1 \quad (22)$$

and value function;

$$V^*(w) = \frac{e^{-\frac{a}{b}w} - e^{-\frac{a}{b}m}}{e^{-\frac{a}{b}n} - e^{-\frac{a}{b}m}}, m \neq n. \quad (23)$$

Proof:

We derive the Hamilton-Jacobi-Bellman (HJB) partial differential equation starting with the Bellman equation:

$$V(w,t;T) = Max_{\pi}\, E[V(w',t+\Delta t;T)] \quad (24)$$

where w', denotes the wealth of the insurer at time $t + \Delta t$.

Rewriting equation (24) as,

$$Max_{\pi}\, E[V(w',t+\Delta t;T) - V(w,t;T)] = 0,$$

and dividing both sides of the equation by Δt and taking limit as Δt tends to zero, the Bellman equation becomes;

$$Max_{\pi}\frac{1}{dt}E[dV] = 0 \quad (25)$$

Ito's lemma (Miao, [26]), which states that,

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial w}dw + \frac{1}{2}\frac{\partial^2 V}{\partial w^2}(dw)^2. \quad (26)$$

For the insurer, substituting in the Ito's lemma for $dW_I^{\pi}(t)$ and $< dW_I^{\pi}(t) >$ using equations (14) and (16), we obtain differential stochastic equation (SDE):

$$\begin{aligned} dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial W_I}\Big[&W_I(t)(\alpha+\beta t) \\ &+ (\mu-\alpha-\beta t)\,\pi_I(t) + \left(\theta - \eta p(t)\right)a \\ &+ \pi_I(t)\beta dZ^{(2)}(t) \\ &+ b\left(1-p(t)\right)d\,Z^{(1)}(t)\Big]dt \\ &+ \frac{\partial^2 V}{2\partial W_I^{\,2}}\Big\{(\pi_I(t)^2\beta^2 + b^2\left(1-p(t)\right)^2 \\ &+ 2\rho\beta b\left(1-p(t)\right)\pi_I(t))\Big\}dt. \end{aligned}$$

(27)

Applying (27) to the Bellman equation (25) and taking expectation, we get the HJB equation;

$$\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial W_I}\left[W_I(t)(\alpha+\beta t) + (\mu-\alpha-\beta t)\,\pi_I(t) + \theta - \eta p t a + \partial 2V 2\partial WI2(\pi It2\beta 2 + b21 - pt2 + 2\rho\beta b1 - pt\pi It\right) = 0, \quad (28)$$

where,

$$E\left(dB_t^{(1)}\right) = E\left(dB_t^{(2)}\right) = 0, \quad (29)$$

satisfying the terminal condition,

$$V(w,T;T) = \frac{W_I^{\,1-\phi}}{1-\phi}\ . \quad (30)$$

Observing the homogeneity of the objective function, the restriction and the terminal condition, we conjecture that the value function V must be linear to $\frac{W_I^{\,1-\phi}}{1-\phi}$.

Let

$$V(w,t;T) = g(t;T)\frac{W_I^{\,1-\phi}}{1-\phi} \quad (31)$$

be such a value function , such that at the terminal date, T

$$V(w,T;T) = g(T;T)\frac{W_I^{\,1-\phi}}{1-\phi}, \quad (32)$$

then,

$$\frac{\partial V}{\partial t} = \frac{W_I^{\,1-\phi}}{1-\phi}g'; \quad \frac{\partial V}{\partial w} = w^{-\phi}g;$$

$$\frac{\partial^2 V}{\partial w^2} = -\phi W^{-1-\phi}g. \quad (33)$$

Substituting equation (33) into equation (28), we obtain; the new H-J-B equation,

$$\frac{W_I^{\,1-\phi}}{1-\phi}g' + \left\{\begin{aligned} &\left[\left[W_I(t)(\alpha+\beta t) + (\mu-\alpha-\beta t)\,\pi_I(t) + \left(\theta-\eta p(t)\right)a\right]\right]W_I^{\,-\phi}g \\ &\frac{-\phi}{2}\left[(\pi_I(t)^2\beta^2 + b^2\left(1-p(t)\right)^2 2\rho\beta b\left(1-p(t)\right)\pi_I(t)\right]W_I^{\,-1-\phi}g \end{aligned}\right\} = 0. \quad (34)$$

Now assuming the insurance company is trying to maximize the probability of beating a given benchmark by some percentage before going below it by another percentage, this objective is related to the case of a manager who stands to receive a bonus achieving the benchmark by a predetermined percentage.

The formalization of this problem takes the form; let $V^*(w)$ denote the maximal probability of beating the benchmark when starting from state w before being beaten by it. That is, let $W_0 = w$ and let p *and* q be given constants with $p < w < q$ such that;

$$V^*(w) = Max_\pi P_w(\tau_m^\pi < \tau_n^\pi). \quad (35)$$

The HJB equation (36); will now be subject to the boundary conditions;

$$V(p) = 0 \text{ and } V(q) = 1 \quad (36)$$

for $m < w < n$.

Since $V(w)$ in this case is independent of time, equation (35) above reduces to;

$$\left[\left[W_I\alpha + (\mu-\alpha)\pi_I(t) + (\theta - \eta p(t))a\right]\right] W_I^{-\phi} g - \frac{\phi}{2}\left[(\pi_I(t)^2\beta^2 + b^2(1-p(t))^2 2\rho\beta b(1-p(t))\pi_I(t)\right] W_I^{-1-\phi} g = 0 \quad (37)$$

for $m < w < n$.

To obtain the optimal value $\pi_I^*(t)$ of $\pi_I(t)$, we differentiate (37), with respect to $\pi_I(t)$ and evaluate to obtain,

$$[(\mu-\alpha)]W_I^{-\phi} g - \phi[\pi_I\beta^2 + b(1-p)\rho\beta]W_I^{-1-\gamma} g = 0. \quad (38)$$

This simplifies to;

$$\pi_I^*(w) = \frac{[(\mu-\alpha)]W_I}{\phi\beta^2} - \frac{\rho b(1-p)}{\beta}. \quad (39)$$

This is the insurer's optimal investment in the risky asset, stock, and this is dependent on the wealth at hand.

Also, differentiating equation (37) with respect to p and simplifying gives the optimal proportion of the reinsured as;

$$W_I^{-\phi} g(-\eta a) + \frac{-\phi W_I^{-1-\phi} g\{-2b^2(1-p) - 2\beta\rho b\,\pi_I\}}{2} = 0, \quad (40)$$

which reduces to;

$$p^* = 1 - \left[\frac{(b^2 + \beta\rho b\,\pi_I)\phi}{\eta a W_I}\right]; 1 > \frac{(b^2 + \beta\rho b\,\pi_I)\phi}{\eta a W_I} > 0. \quad (41)$$

The solution of the HJB equation (29) is thus; replacing π_I and p with their corresponding optimal values π_I^* and p^* as in (39) and (41) respectively and rearranging, we obtain;

$$\frac{\partial V}{\partial W_I}\left[W_I(t)\alpha + (\mu-\alpha)\pi_I^* + (\theta - \eta p^*)a\right] + \frac{\partial^2 V}{2\partial W_I^2}\left[(\pi_I^{*2}\beta^2 + b^2(1-p^*)^2 + 2\rho\beta b(1-p^*)\pi_I^*)\right] = 0 \quad (42)$$

From which we obtain, to get a second order differential equation of the form;

$$A\frac{d^2V}{dw^2} + B\frac{dV}{dw} = 0, \quad (43)$$

where

$$\left.\begin{array}{c} A = \left[(\pi_I^{*2}\beta^2 + b^2(1-p^*)^2 + 2\rho\beta b(1-p^*)\pi_I^*)\right] \\ and \\ B = \left[W_I(t)\alpha + (\mu-\alpha)\pi_I^* + (\theta-\eta p^*)a\right] \end{array}\right\}. \quad (44)$$

The auxiliary equation of (42) is;

$$A\zeta^2 + B\zeta = 0. \quad (45)$$

Solving (44) above, gives

$$\zeta = 0 \text{ or } -\frac{B}{A}, \quad (46)$$

and

$$V(w) = K + Pe^{\frac{B}{A}w}. \quad (47)$$

Applying the boundary conditions, we get;

$$\left.\begin{array}{c} V(p) = K + Pe^{-\frac{B}{A}p} = 0 \\ V(q) = K + Pe^{-\frac{B}{A}q} = 1 \end{array}\right\}. \quad (48)$$

The solution of (48), gives;

$$P = \frac{1}{e^{-\frac{B}{A}q} - e^{-\frac{B}{A}p}} \text{ and } K = \frac{-e^{-\frac{B}{A}p}}{e^{-\frac{B}{A}q} - e^{-\frac{B}{A}p}}. \quad (49)$$

Therefore, the optimal value function is given as;

$$V^*(w) = \frac{e^{-\frac{a}{b}w} - e^{-\frac{a}{b}m}}{e^{-\frac{a}{b}n} - e^{-\frac{a}{b}m}} \; m \neq n \quad (50)$$

which satisfies the boundary conditions.

The case of the Reinsurer:

For the reinsurer, we have; we state the following theorem 2.

Theorem 2: The optimal policy to maximize the expected power utility at T is to invest at each time $t \leq T$;

$$\pi_R^*(t) = \frac{[\mu-\alpha]w_R}{\beta^2\phi} - \frac{b\rho p(t)}{\beta} \quad (51)$$

with optimal proportion reinsured,

$$p^*(t) = 1 - \left[\frac{\beta\rho\, b\pi_r(t)}{a\eta W_R}\right] 0 < \frac{\beta\rho\, b\pi_r(t)}{a\eta W_R} < 1 \quad (52)$$

and value function;

$$V^*(w) = \frac{e^{-\frac{D}{C}w} - e^{-\frac{D}{C}m}}{e^{-\frac{D}{C}n} - e^{-\frac{D}{C}m}}, \quad m \neq n. \quad (53)$$

Proof:

Adopting equations (24) to (34) and replacing $\pi_I(t), W_I(t)$, and $dW_I(t)$ with their respective equivalents, $\pi_R(t)$, $W_R(t)$, and $dW_R(t)$ in equation (35) we obtain the HJB equation;

$$\frac{W_R^{1-\phi}}{1-\phi} g' + \left\{\begin{array}{c} W_R^{-\phi}\left[\left[W_R(t)\alpha + (\mu-\alpha)\pi_R(t) + \eta p(t)a\right]\right] \\ \frac{-\phi W_R^{-1-\phi}}{2}\left[\pi_R^2(t)\beta^2 + b^2p^2(t) + 2\rho\beta b p(t)\pi_R(t)\right] \end{array}\right\} g = 0, \quad (54)$$

which reduces to;

$$g' + \left\{\begin{array}{c} (1-\phi)W_R^{-1}\left[\left[W_R(t)\alpha + (\mu-\alpha)\pi_R(t) + \eta p(t)a\right]\right] \\ \frac{-\phi(1-\phi)W_R^{-2}}{2}\left[\pi_R^2(t)\beta^2 + b^2p^2(t) + 2\rho\beta b p(t)\pi_R(t)\right] \end{array}\right\} g = 0. \quad (55)$$

To obtain the optimal investment in the risky asset, equation (55) is differentiated with respect to $\pi_R(t)$, thus

$$(\mu-\alpha) - \phi W_R^{-1}[\pi_R(t)\beta^2 + \rho\beta b p(t)] = 0. \quad (56)$$

Solving for $\pi_R(t)$ in equation (57) gives the required optimal value;

$$\pi_R^*(t) = \frac{(\mu-\alpha)W_R(t)}{\phi\beta^2} - \frac{\rho b p(t)}{\beta}. \quad (57)$$

The differentiation of (56) with respect to $p(t)$ and simplifying gives the optimal proportion reinsured as;

$$p^*(t) = 1 - \frac{\beta\rho b\pi(t)}{\eta a W_R}; 0 < \frac{\beta\rho b\pi(t)}{\eta a W_R} < 1 \quad (58)$$

We have the replica of equation (42) for the reinsurer as;

$$\frac{\partial V}{\partial W_R}\left[W_R(t)\alpha + (\mu-\alpha)\pi_R^* + \eta p^* a\right] + \frac{\partial^2 V}{2\partial W_R^2}\left[(\pi_R^{*2}\beta^2 + b^2p^{*2} + 2\rho\beta b p^*\pi_R^*\right] = 0, \quad (59)$$

which reduces to;

$$C\frac{d^2V}{dw^2} + D\frac{dV}{dw} = 0, \quad (60)$$

where

$$\left.\begin{matrix} C = \left[(\pi_R{}^{*2}\beta^2 + b^2p^{*2} + 2\rho\beta bp^*\pi_R{}^*\right] \\ and \\ D = [W_R(t)\alpha + (\mu - \alpha)\,\pi_R{}^* + \eta p^* a] \end{matrix}\right\}. \quad (61)$$

The auxiliary equation of (60) is;

$$C\xi^2 + D\xi = 0\,. \quad (62)$$

Solving (62) above, gives

$$\xi = 0 \ or \ -\frac{D}{C}, \quad (63)$$

and

$$V(w) = K + Pe^{\frac{-D}{C}w}\,. \quad (64)$$

Applying the boundary conditions, we get;

$$\left.\begin{matrix} V(p) = K + Pe^{\frac{-D}{C}m} = 0 \\ V(q) = K + Pe^{\frac{-D}{C}n} = 1 \end{matrix}\right\}. \quad (65)$$

The solution of (48), gives;

$$P = \frac{1}{e^{-\frac{D}{C}n} - e^{\frac{-D}{C}m}} \ and \ K = \frac{-e^{-\frac{D}{C}p}}{e^{-\frac{D}{C}n} - e^{-\frac{D}{C}m}}. \quad (66)$$

Therefore, the optimal value function is given as;

$$V^*(w) = \frac{e^{-\frac{D}{C}w} - e^{-\frac{D}{C}m}}{e^{-\frac{D}{C}n} - e^{-\frac{D}{C}m}}, m \neq n \quad (67)$$

which satisfies the boundary conditions.

IV. The Equality of the Insurer's and the Reinsurer's Strategies

The condition under which the proportion reinsured by the Insurer equals the amount accepted to be insured by the Reinsurer is that;

$$p^* = 1 - \left[\frac{(b^2 + \beta\rho b\,\pi_I)\phi}{\eta a W_I}\right] = 1 - \frac{\beta\rho b \pi_R(t)}{\eta a W_R}. \quad (68)$$

This reduces to;

$$\frac{\pi_R(t)}{W_R} = \frac{(1+k)\,\pi_I(t)\phi}{W_I}. \quad (69a)$$

where;

$$b = k\rho\beta\pi_I(t), k > 0\,. \quad (69b)$$

Therefore,

$$\phi = \frac{w_r}{(1+k)w_i}, \quad (70)$$

where, w_r and w_i are the Reinsurer's and the Insurer's portfolio weights in the risky asset, respectively.

V. Conclusion

In this study, we consider the optimal probability of survival investment problem for both an insurer and a reinsurer. The basic claim process is assumed to follow a Brownian motion with drift and the Insurer could purchase proportional reinsurance from the Reinsurer.

The Reinsurer and the Insurer were allowed to invest in a risky and a risk-free assets and expressions for their optimal portfolios obtained solving the corresponding HJB equations. The discount value, ϕ, that would warrant reinsurance, according to the optimal reinsurance proportion chosen by the insurer was obtained.

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