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Oscillation Criteria for a Class of Second Order Neutral Equations

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***Abstract* – In this paper, a class of seconder order neutral equations with distributed deviating arguments are investigated. Several**

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I. Introduction

There have been some results on the oscillatory and asymptotic behavior of second order neutral equations. Here we particularly refer the reader to the papers [1-4]. The purpose of this paper is to establish some oscillation theorems for the following second order neutral equations with distributed deviating arguments

$$(x(t)+\int_a^b p(t,\eta)x[h(t,\eta)]d\rho(\eta))'' \qquad (1)$$

$$+\int_c^d q(t,\xi)f(t,\xi,x[g(t,\xi)])d\sigma(\xi)=0. \quad t\ge t_0$$

Throughout this paper, we assume that the following conditions hold:

H1) $p(t,\eta)\in C([t_0,\infty)\times[a,b],R), \quad p(t,\eta)\ge 0$;

H2) $h(t,\eta)\in C([t_0,\infty)\times[a,b],R), \quad h(t,\eta)\le t,$

$\lim_{t\to\infty}\min_{\eta\in[a,b]} h(t,\eta)=\infty$;

H3) $q(t,\xi)\in C([t_0,\infty)\times[c,d],R_+), \quad q(t,\eta)>0$;

H4) $g(t,\xi)\in C([t_0,\infty)\times[c,d],R), \lim_{t\to\infty}\min_{\xi\in[c,d]} g(t,\xi)=\infty$;

$g(t,\xi)\le t$ is nondecreasing with respect to ξ, $\frac{d}{dt}g(t,c)$ exists and $g'(t,c)>0$;

H5) $\rho(\eta)\in C([a,b],R), \quad \sigma(\xi)\in C([c,d],R)$ are nondecreasing,and the integral of (1) is a Stieltjes one.

In the sequel, it will be always assumed that solutions of (1) exist for any $t_0>0$. A solution $x(t)$ of (1) is called eventually positive solution (or negative solution), if there exists a sufficiently large positive number $t_1\ge t_0$ such that $x(t)>0$ (or $x(t)<0$) for all $t\ge t_1$. A nontrivial solution $x(t)$ of (1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

We say that a function $H=H(t,s)$ belongs to a function class E, denoted by $H\in E$, if $H\in C(D,R^+)$ satisfies

(I) $H(t,t)=0, \quad H(s,t)>0, \quad t>s$;

(II) partial derivatives $\frac{\partial H}{\partial s}$ exist and is continuous, $H_s'(t,s)\le 0$ and there exists $h(t,s)\in C(D,R)$ such that

$$\frac{\partial H}{\partial s}=h(t,s)H(t,s),$$

where $D=\{(t,s): -\infty<s\le t<\infty\}$.

Let $z\in C(R,R)$, we define integral operator

$$Y_{T,t}^H=\int_T^t H(t,s)z(s)ds \qquad (2)$$

for $t\ge T\ge t_0$.

II. The Main Result

Theorem 2.1. Assume that $H\in E$, and there exist function $Q(t,\xi)\in C([t_0,\infty)\times[a,b],R_+)$ which is not eventually zero on any ray $[t_\mu,\infty)\times[a,b]$, and $F(x)\in C(R,R)$ such that

$$f(t,\xi,x)\operatorname{sgn}x\ge Q(t,\xi)F(x)\operatorname{sgn}x, \qquad (3)$$

$$-F(-x)\ge F(x)\ge \lambda x>0, (x>0, \lambda>0 \text{ is a constant}) \qquad (4)$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}Y_{t_0,t}^H[K(s)-\tfrac{1}{4}\tfrac{h^2(t,s)}{g'(s,c)}]=\infty, \qquad (5)$$

Then (1) is oscillatory, in which

$$K(s)=\int_c^d \lambda q(t,\xi)Q(t,\xi)\{1-P[g(t,\xi)]\}d\sigma(\xi),$$

$$P(t)=\int_a^b p(t,\eta)d\rho(\eta).$$

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of(1) on $[t_0,\infty)$ such that $x(t)\ne 0$. Without loss of generality, assume that $x(t)>0, t\ge t_0$. From (H2) and (H5), there exists a $t_1\ge t_0$ such that

$$x(t)>0,\ x[h(t,\eta)]>0,\ x[g(t,\xi)]>0,\ t\ge t_1. \qquad (6)$$

Let

$$y(t)=x(t)+\int_a^b p(t,\eta)x[h(t,\eta)]d\rho(\eta), \quad t\ge t_1 \qquad (7)$$

Then we have

$$y''(t)+\int_c^d q(t,\xi)f(t,\xi,x[g(t,\xi)])d\sigma(\xi)=0. \quad t\ge t_1 \qquad (8)$$

Furthermore, we have $y(t)\ge x(t)>0,\ y''(t)\le 0$ for $t\ge t_1$. We can prove $y'(t)\ge 0, t\ge t_1$. In fact, if it is not true, then there exists a $t_2\ge t_1$ such that $y'(t_2)<0$. Because $y'(t)$ Is decreasing, there exists a $t_3\ge t_2$ such that $y'(t_3)<0$, and

$y'(t)\le y'(t_3)<0, t\ge t_3$.

Integral from t_3 to t, we get

$y(t)\le y(t_3)+y'(t_3)(t-t_3)$.

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When $t \to \infty$, $y(t) \to -\infty$, which is conflict to $y(t) > 0$.

From (7), we have

$$\begin{aligned} x(t) &= y(t) - \int_a^b p(t,\eta) x[h(t,\eta)] d\rho(\eta) \\ &\geq y(t) - \int_a^b p(t,\eta) y[h(t,\eta)] d\rho(\eta) \\ &\geq y(t) - \int_a^b p(t,\eta) y(t) d\rho(\eta) \\ &= [1 - P(t)] y(t). \end{aligned} \quad (9)$$

From (1) (3) (4) and (9), we have

$$y''(t) + \int_c^d \lambda q(t,\xi) Q(t,\xi)\{1 - P[g(t,\xi)]\} y[g(t,\xi)] d\sigma(\xi) \leq 0.$$

As $g(t,\xi)$ is nondecreasing with respect to ξ, we have

$$y''(t) + y[g(t,c)] \int_c^d \lambda q(t,\xi) Q(t,\xi)\{1 - P[g(t,\xi)]\} d\sigma(\xi) \leq 0.$$

Let

$$z(t) = \frac{y'(t)}{y[g(t,c)]},$$

then

$$\begin{aligned} z'(t) &= \frac{y''(t)}{y[g(t,c)]} - \frac{y'(t) y'[g(t,c)] g'(t,c)}{y^2[g(t,c)]} \\ &\leq -\int_c^d \lambda q(t,\xi) Q(t,\xi)\{1 - P[g(t,\xi)]\} d\sigma(\xi) - g'(t,c) z^2(t) \\ &= -K(t) - g'(t,c) z^2(t), \end{aligned}$$

i.e.

$$K(t) \leq -z'(t) - g'(t,c) z^2(t). \quad (10)$$

Using operator $Y_{T,t}^H$ for (10), we have

$$\begin{aligned} Y_{T,t}^H(K) &\leq -\int_T^t H(t,s) z'(s) ds - \int_T^t H(t,s) g'(s,c) z^2(s) ds \\ &= z(T) H(t,T) + \int_T^t H(t,s) h(t,s) z(s) ds \\ &\quad - \int_T^t H(t,s) g'(s,c) z^2(s) ds \\ &= z(T) H(t,T) - Y_{T,t}^H [h(t,s) z(s) + g'(s,c) z^2(s)] \\ &= z(T) H(t,T) - Y_{T,t}^H \{g'(s,c)[z(s) + \tfrac{1}{2} \tfrac{h(t,s)}{g'(s,c)}]^2\} \\ &\quad + \tfrac{1}{4} Y_{T,t}^H [\tfrac{h^2(t,s)}{g'(s,c)}], \end{aligned}$$

So

$$\begin{aligned} & Y_{T,t}^H [K(s) - \tfrac{1}{4} \tfrac{h^2(t,s)}{g'(s,c)}] \\ &\leq z(T) H(t,T) - Y_{T,t}^H \{g'(s,c)[z(s) + \tfrac{1}{2} \tfrac{h(t,s)}{g'(s,c)}]^2\}. \end{aligned} \quad (11)$$

For $t \geq t_2$, we have

$$Y_{t_2,t}^H [K(s) - \tfrac{1}{4} \tfrac{h^2(t,s)}{g'(s,c)}] \leq z(t_2) H(t,t_2) \leq H(t,t_0) | z(t_2) |.$$

Furthermore, we have

$$\begin{aligned} & Y_{t_0,t}^H [K(s) - \tfrac{1}{4} \tfrac{h^2(t,s)}{g'(s,c)}] \\ &= Y_{t_0,t_2}^H [K(s) - \tfrac{1}{4} \tfrac{h^2(t,s)}{g'(s,c)}] + Y_{t_2,t}^H [K(s) - \tfrac{1}{4} \tfrac{h^2(t,s)}{g'(s,c)}] \\ &\leq Y_{t_0,t}^H [| K(s) |] + | z(t_2) | H(t,t_0) \\ &\leq H(t,t_0) [\int_{t_0}^t | K(s) | ds + | z(t_2) |], \quad t \geq t_2. \end{aligned}$$

Then

$$\frac{1}{H(t,t_0)} Y_{t_0,t}^H [K(s) - \tfrac{1}{4} \tfrac{h^2(t,s)}{g'(s,c)}] \leq \int_{t_0}^t | K(s) | ds + | z(t_2) |, \quad t \geq t_2. \quad (12)$$

Let $t \to \infty$, (12) is contradict with (5). Then Eq.(1.1) is oscillatory.

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