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Classical Versions of Quantum Stochastic Processes Associated with the Oscillator Algebra

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***Abstract* – It has been known for a long time that any infinitely divisible distribution (I.D.D) can be realized on a symmetric Fock space with an appropriate noise space. This realization led to a kind of correspondence between Lie algebras and I.D.D. Namely, each I.D.D (or equivalently Lévy process) leads to a such Lie algebra commutation relations. In this context, it was shown (see [1]) that the the quantum stochastic processes corresponding to the bounded form of the oscillator algebra can not cover a large classes of Lévy processes, in particular the non standard Meixner classes. For this reason, we consider the unbounded form of the oscillator algebra called *the adapted oscillator algebra*. Then, we prove that its Fock representation can give rise to the infinitely divisible processes such as the Gamma, Pascal and the Meixner-Pollaczek.**

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I. Introduction

In the quantum probability theory, a quantum stochastic process (QSP in the following) associated with a given $*$–Lie-algebra may cover infinitely classical processes, i.e., it has many classical versions or sub-processes. Usually, it is not easy to identify these classical processes. Despite the principle "*algebra implies statistics*", the initial algebra determines the statistics of such associated QSP: Concrete examples have shown such difficulties when identifying these versions. The main idea of the *tomography* of QSP is the following:

Giving a QSP associated with a such $*$–Lie algebra, when *restricted to self-adjoint abelian sub-algebras*, what statistical information can be deduced and what is the classical structures arising from QSP?

This idea was supported by the local Kolmogorovian aspect of quantum probability theory, i.e., any QSP has a fine classical structure. Exactly as in differential geometry, every manifold is locally Euclidean (in a topological sense), where global properties are described by local charts. But, here, localization means: restricted to abelian sub-algebras.

Contrarily to the case of the one-mode algebra of the square of white noise $(\equiv sl(2,\mathbb{R}))$, the QSP associated with the Fock representation of the one-mode oscillator algebra give a rise to only Gaussian and Poisson processes. This is still true even in the infinite mode, i.e., when taking the test function's space to be $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ (see [1], [2], [6], [12] for more details). But in connection with the quantum decomposition of classical random variables, the inverse procedure of the tomography (see [4]and [5]), the quantum structure (i.e., global structure in the sense of the tomography) of classical Lévy processes was related to the oscillator algebra with a test functions not necessarily in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$. This led us to modify the test function space in order to reveal a such classical sub-process and highlight statistical information supported by them.

In view of Fock space factorization property, the expected sub-processes are of Lévy type, i.e., their probability laws are infinitely divisible. Then, the classification of the classical copies of the QSP will be based on the Lévy exponent.

This paper is organized as the following: Section 2 is a preliminary which can be read diagonally by specialists in quantum theory. In this section, we give a detailed background on the notion of bosonic Fock space. We also recall important notions and results around fundamental operators as well as the creation, annihilation and conservation operators. Next, we give the Lévy-kinchine formula which help us to identify such sub-processes. In the third section, we introduce the notion of adapted oscillator algebra w.r.t such space of test functions and its corresponding QSP. Then, we bring out the involved classical sub-processes and we compute theirs characteristic exponents. In Section 4, we identify such copies via theirs Lévy exponents and we demonstrate that many classes of Lévy processes can be sub-processes of

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the initial quantum one. That is many Lévy processes arise from tomography of QSP associated with the oscillator algebra.

II. Preliminaries

2.1. Background on the Fock Space.

Let $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}, dx)$ be the complex Hilbert space of the $\mathbb{C}$ – valued, square integrable functions on R with scalar product, denoted by $\langle .,. \rangle$ which is linear in the second variable.

We denote

$$\mathcal{H}^{\circ n} = \underbrace{\mathcal{H} \circ \cdots \circ \mathcal{H}}_{n \text{ times}}$$

the *n*-fold symmetric tensor product of $\mathcal{H}$, when for n=0, it is identified to the one dimensional space $\mathcal{H} := \mathbb{C}\Phi$, where Φ is a unital vector and one calls it vacuum vector. The symmetric or bosonic Fock space over $\mathcal{H}$ is defined by

$$\Gamma(\mathcal{H}) := \bigoplus_{n=0}^{+\infty} \mathcal{H}^{on} \quad (1)$$

An element f of $\Gamma_s(\mathcal{H})$ is a sequence $f = (f_n)_{n\geq 0}$ with $f_n \in \mathcal{H}^{on}$ and also we write $f = \sum_{n\geq 0} f_n$. The scalar product, on $\Gamma_s(\mathcal{H})$, and the associated norm are given by

$$\langle f, g \rangle = \sum_{n\geq 0} \langle f_n, g_n \rangle, \; f = (f_n)_{n\geq 0}, g = (g_n)_{n\geq 0} \in \Gamma_s(\mathcal{H})$$

and

$$\|f\|^2 = \sum_{n\geq 0} \|f\|^2, \quad f = (f_n)_{n\geq 0} \in \Gamma_s(\mathcal{H})$$

We define ψ_f, the *exponential* (*coherent*) vector in $\Gamma_s(\mathcal{H})$ associated with $f \in \mathcal{H}$ by

$$\psi_f := \sum_{n\geq 0} \frac{f^{on}}{\sqrt{n!}}, \quad \psi_0 = \Phi$$

The set of exponential vectors $\{\psi_f, f \in \mathcal{H}\}$ is total and linearly independent in $\Gamma_s(\mathcal{H})$. Moreover, we have

$$\langle \psi_f, \psi_g \rangle = e^{\langle f, g \rangle}, \; \forall f, g \in \mathcal{H}$$

Denoting ε, the subspace of $\Gamma_s(\mathcal{H})$, generated by the set of the exponential vectors. Then the *bosonic creation* and *annihilation* operators are densely defined on the Fock space $\Gamma_s(\mathcal{H})$, by their actions on ε as follows:

$$A(\varphi)\psi_f := \langle \varphi, f \rangle \psi_f ; A^+(\varphi)\psi_f := \frac{d}{ds}\Big|_{s=0} \psi_f + s\varphi$$

Notice that $A^+(\varphi)$ is linear in φ, but $A^-(\varphi)$ is anti-linear.

It is well-known (see [9] and [8]) that the operators $A^+(\varphi)$ and $A^-(\varphi)$ are closable (they have a densely defined adjoint). We extend them by closure, while keeping the same notations $A^+(\varphi)$, $A^-(\varphi)$, they are mutually adjoint. Moreover, they satisfy the *canonical commutations relations* (CCR):

$$\left[A^-(\varphi), A^+(\phi)\right] = \langle \varphi, \phi \rangle 1, \quad (2)$$

$$\left[A^-(\varphi), A^-(\phi)\right] = \left[A^+(\varphi), A^+(\phi)\right] = 0, \quad (3)$$

for any $\phi, \varphi \in \mathcal{H}$, where $[x, y] := xy - yx$ is the commutator and **1** is the identity operator on $\Gamma_s(\mathcal{H})$.

Given an unitary operator U of $\mathcal{H}$, it is possible to rise, in a natural way, this operator into unitary operator, $\Gamma(U)$ of $\Gamma_s(\mathcal{H})$ by putting $\Gamma(U) := \bigotimes_{k=1}^{n} U$, when restricted to $\mathcal{H}^{on}$ for $n \geq 1$, and equals to identity, on $\mathcal{H}_0$. This operator $\Gamma(U)$ is called the *second quantized* of U and it is easy to check that

$$\Gamma(UV^*) = \Gamma(U)\,\Gamma(V)^*$$

If $\phi : \mathbb{R} \to \mathbb{R}$ is a real valued function, then the associated multiplication operator M_ϕ (which is not necessarily bounded) is self-adjoint. The *differential second quantized* $\Lambda(\phi)$ (or the *conservation operator*) of the self-adjoint operator M_ϕ of $\mathcal{H}$ is defined via the Stone theorem (see [10]) by

$$\Gamma\left(e^{itM_\phi}\right) =: e^{it\Lambda(\phi)}, \qquad t \in \mathbb{R}$$

When $\phi : \mathbb{R} \to \mathbb{C}$ is a complex valued function, we decompose it as a complex sum of two real valued functions $\phi = \phi_1 + i\phi_2$. We define its conservation operator by

$$\Lambda(\phi) := \Lambda(\phi_1) + i\Lambda(\phi_2)$$

Hence $\Lambda(\phi)$ is linear in ϕ.

The second quantized operator of U acts on ε by

$$\Gamma(U)\psi_f := \psi Uf$$

It follows that

$$\Lambda(\varphi)\psi_f = \frac{d}{ds}\Big|_{s=0} \psi_{e^{s\phi}f} = A^+(\phi f)\psi_f$$

It is well-known that the operators $\varepsilon^{A^+(\varphi)}$ and $\varepsilon^{A^-(\varphi)}$ for $\varphi \in \mathcal{H}$, are well-defined on the set of the exponential vectors E by their series. Moreover, they act on this set as follows:

$$e^{A^+(\varphi)}\psi_f = \psi_{f+\varphi}; \; e^{A^-(\varphi)}\psi_f = e^{\langle \varphi, f \rangle}\psi_f, \; f \in \mathcal{H}$$

The following canonical commutation relations hold weakly on the set of the exponential vectors

$$\left[\Lambda(\phi), A^+(\varphi)\right] = A^+(\phi\varphi); \left[\Lambda(\phi), A(\varphi)\right] = -A(\bar{\phi}\varphi);$$

$$\left[\Lambda(\phi_1), \Lambda(\phi_2)\right] = 0$$

2.2 Lévy Processes and Infinitely Divisible Laws

It is known (see [11] and [7]), that any *infinitely divisible* probability measure μ on $\mathbb{R}$ is canonically associated with a triple (α,σ,β) such that:

– α is a real constant

– β is a positive finite measure on $\mathbb{R}$ with

$$\sigma^2 = \beta(\{0\})$$

– denoting $\hat{\mu}$ the Fourier transform of μ and Ψ, the function

$$\Psi(x) = i\alpha x - \frac{\sigma^2}{2}x^2 + \int_{\mathbb{R}\setminus\{0\}}\left(e^{ixt} - 1 - \frac{ixt}{1+t^2}\right)\nu(dt);\ x\in\mathbb{R}, \tag{4}$$

where

$$\nu(dt) = \frac{1+t^2}{t^2}d\beta(t), \quad t\in\mathbb{R}\setminus\{0\} \tag{5}$$

one has

$$\hat{\mu}(x) =: e^{\Psi(x)}; \quad x\in\mathbb{R} \tag{6}$$

Conversely, given any such a triple (α,σ,β), there exists an infinitely divisible probability measure on R whose has the form (6) with Ψ given by (4).

The function Ψ is called the *Lévy–Khintchine function*, or the *characteristic exponent*, of μ and the triple (α,σ,β) is called a generating triple for the measure μ. Finally, the measure ν on $\mathbb{R}\setminus\{0\}$ is called the *Lévy measure* of μ.

III. Classical Stochastic Processes Through the Oscillator Algebra

Definition 1: The infinite dimensional oscillator algebra over $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R})$ is the complex $*$–Lie algebra $\mathcal{L}_{osc}(\mathbb{R})$ with linearly independent generators

$$\left\{A^+(\phi), A^-(\varphi), \Lambda(\psi), 1 : \phi,\varphi,\psi\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})\right\}$$

and relations

$$\left[A^-(\varphi), A^+(\phi)\right] = \langle\varphi,\phi\rangle 1, \tag{7}$$

$$\left[\Lambda(\phi), A^\pm(\varphi)\right] = \pm A^\pm(\varphi^\pm\varphi), \tag{8}$$

$$\left[A^\pm(\varphi), A^\pm(\phi)\right] = \left[\Lambda(\varphi), \Lambda(\phi)\right] = 0, \tag{9}$$

$$(A^-)^*(\varphi) = A^+(\varphi), \quad \Lambda^*(\phi) = \Lambda(\bar{\phi}), \tag{10}$$

where **1** is the identity operator on $\mathcal{H}$ with the notation

$$\phi^+ = \phi;\ \phi^- = \bar{\phi}$$

The maps $\phi\mapsto A^+(\phi), \Lambda(\phi)$ are linear in ϕ while $\phi\mapsto A^-(\phi)$ is anti-linear.

The fundamental quantum stochastic processes associated with the oscillator algebra are given by the following relations:

$$A_t^\pm(\phi) = A^\pm(\chi_{[0,t]}\otimes\phi);\ \Lambda_t(\phi) = \Lambda(\chi_{[0,t]}\otimes\phi)$$

Recall that these stochastic processes act on the Hilbert space

$$\Gamma\left(L^2([0,t],\mathcal{H})\right) = \Gamma\left(L^2([0,t]\otimes\mathcal{H})\right)$$

The more general quantum process associated to $\mathcal{L}_{osc}(\mathbb{R})$ is given by the complex linear combination

$$X_t = A_t^+(\varphi_1) + A_t^-(\varphi_2) + \Lambda_t(\phi) + t\lambda 1,$$

where $\varphi_1,\varphi_2,\phi\in L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$ and $\lambda\in\mathbb{C}$.

Denote by $\mathcal{C}$, the family of classical stochastic processes associated with the algebra $\mathcal{L}_{osc}(\mathbb{R})$ (i.e., sub-processes of X_t). It was shown (see [1]) that $\mathcal{C}$ can not recover a large class of infinitely divisible processes. At least, the three non standard Meixner classes (Gamma, Pascal and Meixner-Pollaczek) are not in $\mathcal{C}$. This abstraction is due to the fact that the test functions in the argument of $A^\pm$ are bounded (see Def 1). Our aim is to adapt the oscillator algebra in order to include a large class of infinitely divisible processes. For this goal, we shall extend the test function space at the level of the argument of the conservation operator.

Definition 2. Let $\mathcal{M}$ be a space of test functions $\mathbb{R}\to\mathbb{C}$, closed under complex conjugation and let $\mathcal{S}$ be the sub-space of $\mathcal{H}$ given by

$$\mathcal{S} := \left\{\varphi\in\mathcal{H};\quad \phi_n......\phi_1\varphi\in\mathcal{H}\quad \forall\phi_1,......,\phi_n\in\mathcal{M},\ n\geq 1\right\}$$

The adapted oscillator algebra $\mathcal{L}'_{osc}(\mathbb{R})$ w.r.t the space $\mathcal{M}$ is the complex $*$–Lie algebra generated by the set

$$\left\{A^+(\varphi_1), A^-(\varphi_2), \Lambda(\phi), 1;\ \varphi_i\in\mathcal{S},\ \phi\in\mathcal{M},\quad i=1,2\right\}$$

with relations given as in equations from (7) to (10).

Remark 3

1. Unlike the definition (1), the test functions appearing in the definition (2) are not necessarily bounded. Then, if $\mathcal{M}\neq L^\infty(\mathbb{R})$, the multiplication operators $\mathcal{M}_\phi$ are not bounded for all $\phi\in\mathcal{M}$. For this, $\mathcal{L}'_{osc}(\mathbb{R})$ can be considered as the unbounded form of the oscillator algebra. The transition to the unbounded form will give as more degree of freedom at the level of test functions in order to recover a large class of infinitely divisible distributions.
2. From the definition of $\mathcal{S}$, clearly that (8) make sense.
3. In the definition of S, the condition

$$\left(\varphi\in\mathcal{H};\quad \phi_n.....\phi_1\varphi\in\mathcal{H}\quad \forall\phi_1,......,\phi_n\in\mathcal{M}\right)$$

can be rephrased (with help of the polarization formula) as follows:

$$\phi^n\varphi\in\mathcal{H}\quad \forall\phi\in\mathcal{M}\quad \forall n\in\mathbb{N} \tag{11}$$

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4. The constraint that $\mathcal{M}$ is closed under complex conjugation was given to guarantee the meaning of relation (10).

5. Note that a such paire $(\mathcal{S},\mathcal{M})$ exists, in fact we can take $\mathcal{M}=L^{\infty}_{\mathbb{C}}(\mathbb{R})$, then $\mathcal{S}=\mathcal{H}$. But it was shown in the paper [1] that this choice can not give a rise to Lévy process such the Meixner class. For this reason, $\mathcal{S}$ must be a proper sub-space of $\mathcal{H}$.

Using Equations from (7) to (10), one easily deduces that the quantum stochastic processes A_t^+, A_t^- and Λ_t obey the following relations:

$$\left[A_t^-(\varphi_1), A_t^+(\varphi_2)\right] = t\langle\varphi_1,\varphi_2\rangle 1_0, \tag{12}$$

$$\left[\Lambda_t(\phi), A_t^{\pm}(\varphi)\right] = \pm A_t^{\pm}(\varphi^{\pm}\varphi), \tag{13}$$

$$\left[A_t^{\pm}(\varphi_1), A_t^{\pm}(\varphi_2)\right] = 0 \tag{14}$$

$$\left[\Lambda_t(\phi_1), \Lambda_t(\phi_2)\right] = 0, \tag{15}$$

$$(A_t^-)^*(\varphi) = A_t^+(\varphi);\quad (A_t^*)(\phi) = \Lambda_t(\bar{\phi}) \tag{16}$$

for all $\varphi,\varphi_i\in\mathcal{S}$ and $\phi,\phi_i\in\mathcal{M}$, $i=1,2$.

3.1 Abelian sub-algebras of $\mathcal{L}'_{osc}(\mathbb{R})$ and associated classical processes

One of the basic tenet of quantum probability is that: If $(X_t)_{t\in I}$ is a family of operators acting on the same Fock space L(H) with vacuum vector Ω and $(\tilde{X}_t)_{t\in J}$ is a self-adjoint abelian sub-family of $(X_t)_{t\in I}$, then, under additional analytical conditions which are automatically satisfied in the case we are considering, there exists a classical stochastic process $(Y_t)_{t\in J}$ on a some probability space $(\mathfrak{E},\mathcal{H},\mathbb{P})$ such that, for all bounded complex valued Borel functions $\varphi_1,\varphi_2,\ldots\ldots,\varphi_n$, $n\in\mathbb{N}$ one has

$$\mathbb{E}\left(\varphi_1(Y_{t_1})\varphi_2(Y_{t_2})\ldots\varphi_n(Y_{t_n})\right) = \left\langle\Omega, \varphi_1(\tilde{X}_{t_1})\varphi_2(\tilde{X}_{t_2})\ldots\varphi_n(\tilde{X}_{t_n})\Omega\right\rangle$$

In particular, the characteristic function of $(Y_t)_{t\in J}$ is given by

$$\mathbb{E}(e^{izY_t}) = \left\langle\Omega, e^{iz\tilde{X}_t}\Omega\right\rangle$$

In such case, we say that X_t is a classical sub-process of X_t (or a classical version of X_t).

We apply the above general statement to the case in which the process

$$X_t = A_t^+(\phi) + A_t^-(\varphi) + \Lambda_t(\psi) + t\lambda 1,\quad t\geq 0,$$

acts on the noise space $\Gamma(L^2(\mathbb{R}_+,\mathrm{H}))$. Then such sub-process $(\tilde{X}_t)_{t\geq 0}$, with respect to the vacuum vector, can be identified to an independent increment operator process and its characteristic function is given by

$$\mathbb{E}(e^{iz\tilde{X}_t}) =: e^{i\Psi t(z)},$$

where, in obvious notations, Ψ_t is the cummulant function $\tilde{X}_t$.

For the existence of the classical sub-processes of X_t, we shall find the abelian self-adjoint (AS) sub-algebras of the adapted oscillator algebra $\mathcal{L}'_{osc}(\mathbb{R})$.

Proposition 4

Let $\mathcal{L}'_{osc}(\mathbb{R})$ be an adapted oscillator algebra as in Def 2. Then the abelian self-adjoint sub–algebras take either the form

$$\mathcal{L}_G := Lie\left\{A^+(\varphi) + A^-(\varphi) + \lambda 1;\ \varphi\in\mathcal{K},\ \lambda\in\mathbb{R}\right\} \tag{17}$$

where $\mathcal{K}$ is any real sub-space of $\mathcal{H}$, or the form

$$\mathcal{L}_{ID}^{\xi} := Lie\left\{A^+(\xi\phi) + A^-(\xi\phi) + \Lambda(\phi) + \lambda 1;\ \phi\in\mathcal{K}_{\xi},\ \lambda\in\mathbb{R}\right\} \tag{18}$$

where ξ is a fixed complex valued function and

$$\mathcal{K}_{\xi} := \bigcap_{n\geq 1} L^{2n}_{\mathbb{C}}(\mathbb{R},\eta_{\xi}(dx)) \tag{19}$$

with

$$\eta_{\xi}(dx) = |\xi(x)|^2\, dx$$

Remark 5

1. Note that there are many adapted oscillator algebras and this depends on the choice of test function's space $\mathcal{M}$. Consequently the forms of their sub-algebras also depends on this choice.

2. For the first form (17), we can choose $\mathcal{M}=L^{\infty}_{\mathbb{C}}(\mathbb{R})$, then $\mathcal{K}\subset\mathcal{S}=\mathcal{H}$.

3. For the second form (18), there is many choices of ξ which implies the variety of choice of $\mathcal{M}$. For example the choice $\mathcal{M}=\mathcal{K}_{\xi}$ is convenable which implies that $\mathcal{S}$ contains the sub-space $\xi\mathcal{K}_{\xi}$.

4. Note that the condition $\phi\in\mathcal{K}_{\xi}$ is equivalent to

$$\phi^n\xi\in L^2_{\mathbb{C}}(\mathbb{R}) = \mathcal{H}\qquad \forall n\geq 1. \tag{20}$$

(of Proposition 4).

Let us consider the more general form of elements of $\mathcal{L}'_{osc}(\mathbb{R})$ as:

$$X(\psi,\varphi,\phi,\lambda) = A^+(\psi) + A^-(\varphi) + \Lambda(\phi) + \lambda 1,$$
$$(\psi,\varphi,\phi,\lambda)\in\mathcal{S}^2\times\mathcal{M}\times\mathbb{C}.$$

Then, $X(\psi,\varphi,\phi,\lambda)$ is self-adjoint, if and only if,

$$A^+(\psi) + A^-(\varphi) + \Lambda(\phi) + \lambda 1 = A^-(\psi) + A^+(\varphi) + \Lambda(\bar{\phi}) + \bar{\lambda}1.$$

This gives $\psi=\varphi, \mathfrak{F}(\phi)=0$ and $\lambda\in\mathbb{R}$.

We denote this self-adjoint element by

$$X(\varphi,\phi,\lambda) = A^+(\varphi) + A^+(\varphi) + \Lambda(\phi) + \lambda 1$$

where $\lambda\in\mathbb{R}$ and ϕ is a real valued function in $\mathcal{M}$.

Clearly $X(\varphi,\phi,\lambda)$ belong to an abelian sub-algebra of $\mathcal{L}'_{osc}(\mathbb{R})$, if and only if,

$$\left[X(\varphi,\phi,\lambda),X(\varphi',\phi',\lambda')\right]=0$$
$$\forall\phi,\phi'\in\mathcal{M}_{\phi},\ \varphi,\varphi'\in\mathcal{S}_{\varphi},\lambda,\lambda\in\mathbb{R}$$

where $\mathcal{M}_{\phi}\subset\mathcal{M}$ and $\mathcal{S}_{\varphi}\subset\mathcal{S}$ are two subspaces of $\mathcal{M}$ and $\mathcal{S}$ respectively.

On the other hand, using equations from (7) to (9) and the fact that ϕ and ϕ' are real valued functions, one has

$$\left[X(\varphi,\phi,\lambda),X(\varphi',\phi',\lambda')\right]=$$
$$2i\Im\left(\langle\varphi,\varphi'\rangle\right)1+A^{+}(\phi\varphi'-\phi'\varphi)-A^{-}(\phi\varphi'-\phi'\varphi),$$

which gives

$$\Im\left(\langle\varphi,\varphi'\rangle\right)=0\quad\forall\varphi,\varphi'\in\mathcal{S}_{\varphi} \tag{21}$$

and

$$\varphi\phi'=\phi\varphi'\ \ \forall\phi,\phi'\in\mathcal{M}_{\phi},\ \varphi,\varphi'\in\mathcal{S}_{\varphi}. \tag{22}$$

Note that from Eq.(21), we deduce that the space $\mathcal{S}_{\varphi}$ has a real structure (i.e., the inner product is real valued in the sense that $\langle\varphi,\varphi'\rangle\in\mathbb{R}\,\forall\varphi,\varphi'\in\mathcal{S}_{\varphi}$).

From Eq. (22), we distinguish two cases:

Case. I: $\phi=\phi'=0$

In this case, Eq.(22) becomes trivial. Hence AS-algebra is generated by the elements of the form

$$X(\varphi,\lambda)=A^{+}(\varphi)+A^{+}(\varphi)+\lambda 1,\ \varphi\in\mathcal{K},\ \lambda\in\mathbb{R}.$$

where we have denoted the real sub-space $\mathcal{S}_{\varphi}$ by $\mathcal{K}$.

Case. II: $\phi\neq 0,\phi'\neq 0$

Eq.(22) gives

$$\frac{\varphi}{\phi}=\frac{\varphi'}{\phi'}\quad\forall\phi,\phi'\in\mathcal{M}_{\phi}\quad\forall\varphi,\varphi'\in\mathcal{S}_{\varphi}.$$

Then $\frac{\varphi}{\phi}$ is a constant function. Consequently, there exists a function $\xi:\mathbb{R}\to\mathbb{C}$ such that

$$\varphi=\xi\phi\in L^{2}_{\mathbb{C}}(\mathbb{R})$$

In this case, Eq.(21) is automatically satisfied and AS-algebra is generated by the elements of the form

$$X_{\xi}(\phi,\lambda)=A^{+}_{t}(\xi\phi)+A^{-}_{t}(\xi\phi)+\Lambda_{t}(\phi)+\lambda 1,\ \phi\in\mathcal{M}_{\phi},\ \lambda\in\mathbb{R}.$$

Finally, according to the condition (11), we deduce that

$$\phi^{n}\varphi=\phi^{n+1}\xi\in\mathcal{H}\qquad\forall n\in\mathbb{N},$$

which is equivalent to

$$\phi^{n}\xi\in\mathcal{H}\quad\forall n\geq 1.$$

This gives

$$\int_{\mathbb{R}}|\phi(x)|^{2n}\,|\xi(x)|^{2n}dx<+\infty\ \ \forall_{n}\geq 1.$$

or equivalently $\phi\in L^{2n}_{\mathbb{R}}(\eta_{\xi})\quad\forall_{n}\geq 1.$

Finally $\phi\in\mathcal{K}_{\xi}$ given as in (19).

Corollary 6: The classical sub-processes associated with the adapted oscillator algebra $\mathcal{L}'_{osc}(\mathbb{R})$ take either the form

$$\tilde{X}_{t}=X_{t}(\varphi,\lambda)=A^{+}_{t}(\varphi)+A^{-}_{t}(\varphi)+t\lambda 1 \tag{23}$$

or

$$\tilde{X}_{t}=X^{\xi}_{t}(\phi,\lambda)=A^{+}_{t}(\xi\phi)+A^{-}_{t}(\xi\phi)+\Lambda_{t}(\phi)+t\lambda 1 \tag{24}$$

where the test functions are given as in Proposition (4) and $\lambda\in\mathbb{R}$.

3.2 Characteristic Functions of the Classical Sub-Processes

In this subsection, we investigate to identify these classical processes by computing their characteristic functions in the vacuum state. For this, we need the following lemma.

***Lemma 7** (See [3])*

Let $\mathcal{H}$ be a separable Hilbert space and $\Gamma_{s}\mathcal{H}$ be the bosonic Fock space over $\mathcal{H}$. Let T be a self-adjoint densely defined linear operator on $\mathcal{H}$. Define the functions e_1,e_2 by

$$e_{1}(x):=\frac{e^{x}-1}{x};\ e_{2}(x):=\frac{e^{x}-x-1}{x^{2}},\ x\in\mathbb{C}$$

Let us consider the operators $e_1(iT)$ and of $e_2(iT)$ defined via the spectral theorem. Then for all $u\in\mathcal{H}$ belonging to the domain of $e_1(iT)$ and of $e_2(iT)$, the following identity holds on the domain of the exponential vectors:

$$e^{i\left(A^{+}(u)+A^{-}(u)+\Lambda(T)\right)}=e^{A^{+}(u_{1})}e^{i\Lambda(T)}e^{A^{-}(u_{2})}e^{\gamma} \tag{25}$$

where

$$u_{1}=ie_{1}(iT)u;\ u_{2}=-ie_{1}(-iT)u;\ \gamma=-\langle u,e_{2}(iT)u\rangle \tag{26}$$

Proposition 8: The classical sub-processes $\tilde{X}_t$ given by (23) is a gaussian process with characteristic exponent

$$\Psi(z)=iz\lambda-\|\varphi\|^{2}\frac{z^{2}}{2}.$$

Proof. Since

$$\tilde{X}_{t}=X_{t}(\phi,\lambda)=A^{+}_{t}(\varphi)+A^{-}_{t}(\varphi)+t\lambda 1$$

Then

$$e^{iz\tilde{X}_{t}}=e^{iz(A^{+}_{t}(\varphi)+A^{-}_{t}(\varphi)+t\lambda 1)}$$
$$=e^{i(A^{+}(\chi_{[0,t]}\otimes z\varphi)+A^{-}(\chi_{[0,t]}\otimes(z\varphi))+zt\lambda 1)}$$
$$=e^{i\left(A^{+}(\chi_{[0,t]}\otimes z\varphi)+A^{-}(\chi_{[0,t]}\otimes(z\varphi))\right)}e^{izt\lambda}$$

Setting $u=\chi_{[0,t]}\otimes z\varphi, T=0$. It is clear that all conditions in Lemma (7) are satisfied. Then, one obtains

$$u_{1}=ie_{1}(0)u=iu,\ u_{2}=-ie_{1}(0)u=-iu$$

and

$$\begin{aligned}\gamma &= -\langle u, e_2(0)u\rangle \\ &= -\left\langle \chi_{[0,t]}\otimes(z\varphi), \frac{1}{2}\chi_{[0,t]}\otimes(z\varphi)\right\rangle \\ &= -\left\langle \chi_{[0,t]},\chi_{[0,t]}\right\rangle\left\langle z\varphi,\frac{1}{2}z\varphi\right\rangle \\ &= -t\|\varphi\|^2\frac{z^2}{2}.\end{aligned}$$

Then

$$\begin{aligned}e^{iz\tilde{X}_t} &= e^{A^+(iu)}e^{i\Lambda(0)}e^{A^-(-iu)}e^{\lambda+it\lambda} \\ &= e^{A^+(iu)}e^{A^-(-iu)}e^{\lambda+it\lambda}\end{aligned}$$

This gives

$$\begin{aligned}\mathbb{E}(e^{iz\tilde{X}_t}) &= \left\langle \Omega, e^{iz\tilde{X}_t}\Omega\right\rangle \\ &= \left\langle \Omega, e^{A^+(iu)}e^{A^-(-iu)}e^{\gamma+it\lambda}\,\Omega\right\rangle \\ &= e^{\gamma+it\lambda}\left\langle e^{A^-(iu)}\Omega, e^{A^-(-iu)}\,\Omega\right\rangle \\ &= e^{\gamma+it\lambda}\|\Omega\|^2 \\ &= e^{\gamma+it\lambda} \\ &= e^{t\left(iz\lambda-\|\varphi\|^2\frac{z^2}{2}\right)} \\ &= e^{t\Psi(z)},\end{aligned}$$

where

$$\Psi(z) = iz\lambda - \|\varphi\|^2\frac{z^2}{2}.$$

This completes the proof.

Proposition 9: Let X_t be the classical sub-process given by (25). Then its characteristic function is given by

$$\mathbb{E}(e^{iz\tilde{X}_t}) = e^{t\Psi(z)}, \tag{27}$$

where

$$\Psi(z) = iz\alpha + \int_{\mathbb{R}}\left(e^{iz\phi(x)} - 1 - \frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx), \tag{28}$$

with

$$\alpha = \lambda - \int_{\mathbb{R}}\frac{\phi^3(x)}{1+\phi^2(x)}\eta_\xi(dx) \tag{29}$$

Proof. We will prove Proposition (9) in two steps:

Step.1. Since

$$\tilde{X}_t = X_t^\xi(\phi,\lambda) = A_t^+(\xi\phi) + A_t^-(\xi\phi) + \Lambda_t(\phi) + t\lambda 1$$

Then

$$\begin{aligned}e^{iz\tilde{X}_t} &= e^{iz}\left(A_t^+(\xi\phi) + A_t^-(\xi\phi) + \Lambda_t(\phi) + t\lambda 1\right) \\ &= e^{i(A^+(u)+A^-(\xi\phi)+\Lambda(T))}e^{izt\lambda},\end{aligned}$$

where

$u = \chi_{[0,t]}\otimes(z\xi\phi)$ and $T = M_{\chi_{[0,t]}\otimes(z\phi)} \equiv \chi_{[0,t]}\otimes(z\phi)$

Cleary all conditions in Lemma (7) are satisfied. Then we obtain

$$e^{iz\tilde{X}_t} = e^{A^+}(u_1)e^{i\Lambda(T)}e^{A^-}(u_2)e^{\gamma+iz+\lambda}$$

Note that it is not necessary to explicit u_1 and u_2. In fact, we have

$$\begin{aligned}\mathbb{E}(e^{iz\tilde{X}_t}) &= \left\langle \Omega, e^{iz\tilde{X}_t}\Omega\right\rangle \\ &= \left\langle \Omega, e^{A^+(u_1)}e^{i\Lambda(T)}e^{A^-(u_2)}e^{\gamma+izt\lambda}\,\Omega\right\rangle \\ &= e^{\gamma+izt\lambda}\left\langle e^{A^-(u_1)}\Omega, e^{i\Lambda(T)}e^{A^-(u_2)}\,\Omega\right\rangle \\ &= e^{\gamma+izt\lambda}\left\langle \Omega, e^{i\Lambda(T)}\,\Omega\right\rangle \\ &= e^{\gamma+izt\lambda}\left\langle \Omega,\,\Omega\right\rangle \\ &= e^{\gamma+izt\lambda}.\end{aligned} \tag{30}$$

Now, it is sufficiently to express γ.

Since

$$e_2(iT) = e_2\left(\chi_{[0,t]}\otimes(iz\phi)\right) = \chi_{[0,t]}\otimes e_2(iz\phi).$$

Then from Equation (26), we get

$$\begin{aligned}\gamma &= -\langle u, e_2(iT)u\rangle \\ &= -\left\langle \chi_{[0,t]}\otimes(z\xi\phi), \chi_{[0,t]}\otimes e_2(iz\phi)\left(\chi_{[0,t]}\otimes(z\xi\phi)\right)\right\rangle \\ &= -\left\langle \chi_{[0,t]},\chi_{[0,t]}\right\rangle\left\langle z\xi\phi, e_2(iz\phi)z\xi\phi\right\rangle \\ &= -t\int_{\mathbb{R}}\overline{z\xi(x)\phi(x)}\frac{e^{iz\phi(x)}-iz\phi(x)-1}{(iz\phi(x))^2}z\xi(x)\phi(x)dx \\ &= t\int_{\mathbb{R}}(e^{iz\phi(x)}-iz\phi(x)-1)|\xi(x)|^2\,dx \\ &= t\int_{\mathbb{R}}(e^{iz\phi(x)}-iz\phi(x)-1)\eta_\xi(dx).\end{aligned}$$

Finally , Equation (30) gives

$$\mathbb{E}(e^{iz\tilde{X}_t}) = e^{t\left(iz\lambda+\int_{\mathbb{R}}\left(e^{iz\phi(x)}-iz\phi(x)-1\right)\eta_\xi(dx)\right)} = e^{t\Psi(z)},$$

where $\Psi(z)$ is given by

$$\Psi(z) = iz\lambda + \int_{\mathbb{R}}\left(e^{iz\phi(x)} - iz\phi(x) - 1\right)\eta_\xi(dx). \tag{31}$$

Step.2. Using (31), we obtain

$$\begin{aligned}\Psi(x) &= iz\lambda + \int_{\mathbb{R}}\left(e^{iz\phi(x)} - 1 - \frac{iz\phi(x)}{1+\phi^2(x)} - iz\phi(x) + \frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx) \\ &= iz\lambda + \int_{\mathbb{R}}\left(e^{iz\phi(x)} - 1 - \frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx) \\ &\quad + \int_{\mathbb{R}}\left(-iz\phi(x) + \frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx) \\ &= \int_{\mathbb{R}}\left(e^{iz\phi(x)} - 1 - \frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx) \\ &\quad + iz\left(\lambda + \int_{\mathbb{R}}\left(\frac{\phi(x)}{1+\phi^2(x)} - \phi(x)\right)\eta_\xi(dx)\right) \\ &= iz\left(\lambda - \int_{\mathbb{R}}\frac{\phi^3(x)}{1+\phi^2(x)}\eta_\xi(dx)\right)\end{aligned}$$

$$+\int_{\mathbb{R}}\left(e^{iz\phi(x)}-1-\frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx)$$
$$=iz\alpha+\int_{\mathbb{R}}\left(e^{iz\phi(x)}-1-\frac{iz\phi(x)}{1+\phi^2(x)}\right)\eta_\xi(dx).$$

Note that the condition (20) guaranties the existence of the integral in the equation (29). In fact we have

$$\int_{\mathbb{R}}\frac{\phi^3(x)}{1+\phi^2(x)}\eta_\xi(dx)=\int_{\mathbb{R}}\frac{\phi^3(x)}{1+\phi^2(x)}|\xi(x)|^2(dx)$$
$$\leq\int_{\mathbb{R}}|\xi(x)\phi(x)|^2\,dx<+\infty$$

IV. Identification of the Classical Stochastic Processes $X_t^{\xi}(\phi,\lambda)$

In this section, we will show that how the characteristic exponent Ψ in (28) can recover a large class of Lévy processes. In fact, it is sufficient to consider only two classes of Lévy exponent which are fundamental.

As expected, in view of Equation (28), the characteristic exponent Ψ looks like the Lévy exponent, appearing in the Lévy–Khintchine formula (4). Thus, if there exists a paire (ϕ,ξ) satisfying

$$\phi(\eta_\xi)=\nu \tag{32}$$

one can use the measure-image theorem, to obtain the new expression of Ψ as follows:

$$\Psi(z)=iz\alpha+\int_{\mathbb{R}\setminus\{0\}}\left(e^{izu}-1-\frac{izu}{1+u^2}\right)\nu(du), \tag{33}$$

which corresponds to the Lévy characteristic exponent.

We will discuss the choice of (ϕ,ξ) under condition (32), (i.e., giving (33)) where $\phi\in\mathcal{K}_\xi$, then we will illustrate our result by a simple examples.

Recall that the condition $\phi\in\mathcal{K}_\xi$ is equivalent to (20) which becomes as follows:

$$\int_{\mathbb{R}}x^{2n}\nu(dx)<+\infty\quad\forall n\geq 1 \tag{34}$$

This implies that, only Lévy measures having finite moments of order pair will be considered in our study.

On the other hand, since ν is a Lévy measure, then it satisfies

$$\int_{\mathbb{R}}\min\left(1,x^2\right)\nu(dx)<+\infty \tag{35}$$

But this condition is automatically satisfied. In fact

$$\int_{\mathbb{R}}\min\left(1,x^2\right)\nu(dx)=\int_{|x|\leq1}x^2\,\nu(dx)+\int_{|x|\geq1}\nu(dx)$$
$$\leq\int_{|x|\leq1}x^2\,\nu(dx)+\int_{|x|\geq1}x^2\,\nu(dx)$$
$$=\int_{\mathbb{R}}x^2\,\nu(dx),$$

which is finite from (34).

In the following, we consider two classes of Lévy exponent.

Class I. Characteristic exponents with a positive discrete Lévy measure having a support, $Supp(\nu)=\{a_1,a_2,a_3,....\}$ and satisfying (34), i.e., its Lévy measure is given by

$$\nu(dx)=\sum_{k\geq1}\alpha_k\delta_{a_k}(dx),\ \alpha_k>0,\ a_k\in\mathbb{R} \tag{36}$$

with

$$\sum_{k\geq1}\alpha_k a_k^{2n}<+\infty\quad\forall n\geq1 \tag{37}$$

Class II. Characteristic exponents with a positive continuous Lévy measure having a support, the interval (a,b), i.e., its Lévy measure is expressed as follows:

$$\nu(dx)=f(x)\chi_{(a,b)}(x),-\infty\leq a<b\leq+\infty,\ f(x)>0\ \ \forall x\in(a,b) \tag{38}$$

with

$$\int_a^b x^{2n}f(x)dx<+\infty\quad\forall n\geq1 \tag{39}$$

Theorem 10: Let ν be the Lévy measure given as in (36). Then, the functions ξ and ϕ given by

$$\xi(x)=\sum_{k\geq1}^{\infty}\sqrt{\alpha_k}\chi_{[k,k+1[}(x)\ ;\ \phi(x)=\sum_{k\geq1}a_k\chi_{[k,k+1[}(x) \tag{40}$$

satisfy (32). In this case, the classical stochastic process $X_t^{\xi}(\phi,\lambda)$ has the characteristic exponent (33) corresponding to **Class I**.

Proof. To prove (32) for (ξ,ϕ) given by (40), it is sufficient to prove

$$\int_{\mathbb{R}}g\left(\phi(x)\right)\eta_\xi(dx)=\int_{\mathbb{R}}g(y)\nu(dy)$$

for all positive borel function *g*.

Since $g\left(\phi(x)\right)=\sum_{k\geq1}g(a_k)\chi_{[k,k+1[}(x)$. Then

$$\int_{\mathbb{R}}g\left(\phi(x)\right)\eta_\xi(dx)=\sum_{k\geq1}g(a_k)\eta_\xi([k,k+1[)$$
$$=\sum_{k\geq1}g(a_k)\int_k^{k+1}|\xi(x)|^2\,dx$$
$$=\sum_{k\geq1}g(a_k)\int_k^{k+1}\left(\sum_{j\geq1}\sqrt{\alpha_j}\chi_{[j,j+1[}(x)\right)^2dx$$
$$=\sum_{k\geq1}g(a_k)\int_k^{k+1}\left(\sum_{j\geq1}\alpha_j\chi_{[j,j+1[}(x)\right)dx$$
$$=\sum_{k\geq1}g(a_k)\left(\sum_{j\geq1}\alpha_j\int_k^{k+1}\chi_{[j,j+1[}(x)\,dx\right)$$
$$=\sum_{k\geq1}g(a_k)\alpha_k.$$

On the other hand

$$\int_{\mathbb{R}}g(y)\nu(dy)=\int_{\mathbb{R}}g(x)\left(\sum_{k\geq1}\alpha_k\delta_{a_k}(dx)\right)$$
$$=\sum_{k\geq1}\alpha_k g(a_k)$$

which completes the proof.

Note that ϕ belongs to the space $\mathcal{K}_\xi$. In fact condition (37) implies (34) which is equivalent to 20).

Example 11 *(The Negative binomial process).*

The Lévy measure of the negative binomial process, with parameters $c>0$ and $0<p<1$ is the measure defined on $\mathbb{N}^*$ by

$$\nu(dx)=\sum_{k=1}^{+\infty}\frac{c(1-p)^k}{k}\delta_k(dx)$$

In this case

$$\xi(x)=\sum_{k=1}^{+\infty}\sqrt{\frac{c}{k}}(1-p)^{\frac{k}{2}}\chi_{[k,k+1[}(x)$$

and

$$\phi(x)=\sum_{k=1}^{+\infty}\frac{c(1-p)^k}{k}\chi_{[k,k+1[}(x)$$

Theorem 12: Let ν be the Lévy measure given in (38). If the function ξ is chosen such that ϕ satisfies, the differential equation

$$y'=\frac{|\xi(x)|^2}{f(y)}\qquad \forall x\in Supp(\xi)=:I_\xi \tag{41}$$

and $\phi(I_\xi)=(a,b)$, then the paire (ξ,ϕ) satisfies (32). In this case, the classical stochastic processes $X_t^\xi(\phi,\lambda)$ have the characteristic exponent corresponding to the **Class II**.

Proof.: To prove (32) for (ξ,ϕ) given by (41), it is sufficient to prove

$$\int_{\mathbb{R}} g\left(\phi(x)\right)\eta_\xi(dx)=\int_{\mathbb{R}} g(y)\nu(dy)$$

for all positive borel function g.

Without difficulty, using the change of variable $\left(u=\phi(x)\right)$, we get

$$\begin{aligned}\int_{\mathbb{R}} g\left(\phi(x)\right)\eta_\xi(dx)&=\int_{I_\xi} g\left(\phi(x)\right)|\xi(x)|^2\,dx\\&=\int_{I_\xi} g\left(\phi(x)\right)\phi'(x)f\left(\phi(x)\right)dx\\&=\int_{\phi(I_\xi)} g(u)f(u)du\\&=\int_{(a,b)} g(u)f(u)du\\&=\int_{\mathbb{R}} g(u)\nu(du)\end{aligned}$$

where we have used $\phi(I_\xi)=(a,b)$.

Note that ϕ belongs to the space $\mathcal{K}_\xi$. In fact condition (39) implies (34) which is equivalent to (20).

Example 13 *(The Gamma process).*

It is well-known (see [7]) that the Lévy measure of the Gamma Process of order p, is given by

$$\nu(dx)=\frac{e^{-px}}{x}\chi_{[0,+\infty[}(x)dx$$

which implies that $f(x)=\frac{e^{-px}}{x}$ and $(a,b)=]0,+\infty[$.

We will choose a function ξ such that ϕ obeys the differential equation (41) with condition $\phi(I_\xi)=]0,+\infty[$. In our case, the differential equation (41) will be expressed as follows:

$$y'=ye^{py}|\xi(x)|^2,\ x\in I_\xi \tag{42}$$

Let us fixing a function ξ and assuming that it satisfies

$$\int_0^{+\infty}|\xi(t)|^2\,dt=+\infty\ ;\ \lim_{x\to+\infty}\int_x^{+\infty}|\xi(t)|^2\,dt=0$$

Note that a such ξ exists. As example, we can choose

$$\xi(t)=\begin{cases}\frac{1}{t}, & t\geq 0;\\ 0, & t\leq 0.\end{cases} \tag{43}$$

To find a such solution of the differential equation (42), let us introduce the functions F_p given by

$$F_p(x)=-E_1(px),\quad x>0,$$

where

$$E_1(x)=\int_x^{+\infty}\frac{e^{-t}}{t}dt,\quad x>0,$$

is the exponential integral function. We take w_ξ the function given as follows:

$$\omega_\xi(x)=-\int_x^{+\infty}|\xi(t)|^2\,dt\qquad x>0.$$

Consequently, the function ϕ defined by

$$\phi(x)=F_p^{-1}(\omega_\xi(x)),\quad x>0$$

is a solution of the differential equation (42). In fact, we have

$$\begin{aligned}F_p'(x)&=-pE_1'(px)\\&=-p\frac{-e^{-px}}{px}\\&=\frac{e^{-px}}{x}.\end{aligned}$$

Then

$$\begin{aligned}\phi'(x)&=\omega_\xi'(x)\frac{1}{F_p'\left(F_p^{-1}(\omega_\xi(x))\right)}\\&=\omega_\xi'(x)\frac{1}{F_p'\left(\phi(x)\right)}\\&=\omega_\xi'(x)\frac{\phi(x)}{e^{-p\phi(x)}}\\&=\omega_\xi'(x)\phi(x)e^{p\phi(x)}\\&=\phi(x)e^{p\phi(x)}|\xi(x)|^2,\end{aligned}$$

where we have used $w'_\xi(x)=|\xi(x)|^2$.

On the other hand , we have

$$\begin{aligned}\phi(I_\xi) &= \phi\left(]0,+\infty[\right)\\ &= F_p^{-1}\left(w_\xi\left(]0,+\infty[\right)\right)\\ &= F_p^{-1}\left(]0,+\infty[\right)\\ &=]0,+\infty[,\end{aligned}$$

which gives that the paire (ξ,ϕ) satisfies $\phi(I_\xi)=(a,b)$.

Example 14 *(The Meixner process)*

The Lévy measure of the Meixner process $M(\alpha,\beta,\delta)$ is given by

$$\nu(dx)=\frac{\delta e^{\frac{bx}{p}}}{x sh\left(\frac{\pi x}{p}\right)},\quad p,\ \delta>0, -\pi<b<\pi$$

Then, with the same assumptions as in the Gamma case, we easily verify that the corresponding function ϕ satisfies the differential equation

$$y' = y\frac{|\xi(x)|^2}{2\delta}\left(e^{\frac{\pi-b}{p}y}-e^{-\frac{\pi+b}{p}y}\right) \tag{44}$$

Taking $\phi=\phi_1+\phi_2$, where ϕ_1 satisfies

$$y' = y\frac{|\xi(x)|^2}{2\delta}e^{\frac{\pi-b}{p}y}$$

and ϕ_2 satisfies:

$$y' = -y\frac{|\xi(x)|^2}{2\delta}e^{-\frac{\pi+b}{p}y}$$

Then ϕ can be expressed as follows:

$$\phi(x)=F_{p_1}^{-1}\left(\omega_{\xi_1}(x)\right)\chi_{]-\infty,0[}-F_{p_2}^{-1}\left(\omega_{\xi_2}(-x)\right)\chi_{]0,+\infty0[},$$

where

$$p_1=\frac{\pi-b}{p};\quad p_2=\frac{\pi+b}{p},$$

and

$$\xi_1(x)=\frac{\xi(x)}{2\sigma};\quad \xi_2(x)=\frac{\xi(-x)}{2\sigma}$$

Finally, ϕ is a solution of the differential equation (44).

V. Conclusion

Despite the role of the commutation relations, the space of the test functions has played an important role in the determination of the classical processes corresponding to the oscillator algebra. In fact, the transition from bounded to unbounded form of the oscillator algebra gave us more degree of freedom. This led us to recover more classes of Lévy processes.

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