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Some Notes on Self Centered Graphs

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***Abstract* – This paper evaluates the center of a graph and associated problems. We study about the Eccentricity and Average Eccentricity of self centered crisp graphs and derive some of the properties of self-centered graphs.**

***Keywords* – Graphs, Distance in Graphs, Center Of Graphs, Eccentricity of Graphs, Average Eccentricity, Self Centered Graphs.**

I. Introduction

Graph theory may be said to have it's beginning in 1736 when Euler considered the (general case of the) Konigsberg bridge problem: Does there exist a walk crossing each of the seven bridges of Konigsberg exactly once? It took 200 years before the first book on graph theory was written. Since then graph theory has developed into an extensive and popular branch of mathematics, which has been applied to many problems in mathematics, computer science, and other scientific and not so scientific areas. We discuss the concept of average eccentricity and related ideas in this paper.

Average eccentricity is extensively studied by many researchers in Chemistry and Biomedicine. The average eccentricity has been used as a molecular descriptor since 1988. Yun fang and Bo Zhou[7] has studied lower and upper bounds for the average eccentricity in terms of the numbers of vertices and edges, they have also given lower and upper bounds for average eccentricity of trees with fixed diameter, fixed number of pendant vertices and fixed matching number, respectively.

Andreas and Daniele[5] has determined the eccentricity of an arbitrary vertex, the average eccentricity and its standard deviation for all Sierpinski graphs. The average eccentricity is deeply connected with the topological descriptor the eccentric connectivity index, defined as a sum of products of vertex degrees and eccentricities. Aleksander[1] analyzed the extremal properties of the average eccentricity, introducing two graph transformations that increase or decrease eccentricity of G. In this paper we study the relation between the average eccentricity and radius of a graph. A graph is a pair G:(V,E) where V is a finite set and E is a relation on *V*. In this paper we consider undirected connected finite graphs without loops and multiple edges. Standard notations of the graph theory are used. We recall some of them. We denote by *V(G)* the vertex set and by *E(G)* the set of edges of a graph *G*. The symbol |V(G)| is used for cardinality of V(G). The degree of a vertex u∈V is denoted by $deg_G(u)$. The complete graph with n vertices is denoted by K_n. For a set $S \subseteq V(G)$, *G[S]* is the subgraph induced by S. A connected acyclic graph is called a tree.

II. Preliminaries

The following definitions are taken from the book by Harary [3]. Some of these can also be found in [4]. In a graph ***G = (V, E)***, *V* and *E* denote the vertex set and the edge set of *G*, respectively. Agraph ***G = (V, E)*** is trivial, if it has only one vertex, i.e., *V (G) = 1*; otherwise *G* is nontrivial. The graph *G* is the complete graph if, every two vertex in *G* are adjacent. All complete graphs of order ***n*** are isomorphic with each other, and they will be denoted by K_n. The complement of G is the graph$G^C = (V, E^C)$, where $E^C = \{e / e \notin E\}$. The complements of complete graphs are called discrete graphs. In a discrete graph *E(G)* = ϕ. Clearly, all discrete graphs of order ***n*** are isomorphic with each other. A graph *G* is said to be regular, if every vertex of *G* has the same degree. If this degree is equal to ***r***, then *G* is **r** − regular or regular of degree **r**. A sequence $v_1, v_2 \dots, v_k$of vertices in a graph *G* is called a walk, if $\{v_1v_2, v_2v_3, \dots, v_{k-1}v_k\}$are the edges of *G* and *k* is the length of the walk. A walk $\{v_1v_2, v_2v_3, \dots, v_{k-1}v_k\}$is called a path in *G*, if $v_1, v_2 \dots, v_k$are distinct in *G*. A walk$\{v_1v_2, v_2v_3, \dots, v_{k-1}v_k\}$ is called a cycle in *G*, if$\{v_1, v_2, v_3, \dots, v_k\}$ are distinct vertices and$v_1 = v_k$in *G*.

III. Average Eccentricity of a Graph

If ***u, v*** are two vertices of *G*, then any shortest ***u-v*** - path in *G* is called a ***u-v*** geodesic in *G*. Length of a shortest path between ***u*** and ***v***in *G* is called the distance between *u* and *v*. The eccentricity of the vertex *u* is denoted as *e(u)* and is defined as $e(u) = Max_{v \in V}[d(u, v)]$. If a graph *G* is disconnected, then $e(v) = \infty$ for all vertices v∈V. Radius of *G* is defined as $r(G) = Min[e(v): v \in V]$. Diameter of *G* is defined as$d(G) = Max[e(v): v \in V]$.Average eccentricity of G [2] is $avec(G) = \frac{e(v_1)+e(v_2)+\dots+e(v_n)}{n}$, where, $v_1, v_2, v_3, \dots, v_n$are vertices of *G*. If *e(v) = r(G)*, then *G* is called a central vertex of *G* and the graph induced by all central vertices of *G* is called center of *G* and is denoted as *C(G).* A vertex ***v*** of *G* is called an eccentric vertex of *G* [8] if there exists a vertex ***u*** in *G* such that ***d(u, v) = e(u).*** This means that if the vertex ***v*** is farthest from another vertex ***u***, then ***v*** is an eccentric vertex of ***u*** (as well as G) and is denoted as $u^* = v$.

In a graph if all the vertices have the same eccentricity, then *G* is called a self-centered graph. A graph is eccentric if all vertices of *G* are eccentric vertices. Let *G* be a graph. Then $G^2$is that graph whose vertex set $V(G^2) = V(G)$ with *u*, *v* adjacent in G^2 whenever $d(u, v) \leq 2$. Similarly the graphs $G^3, G^4, \dots, G^n$ are defined.

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Sameena and Sunitha [6] has proved that a connected fuzzy graph is self centered if and only if for every pair of vertices u,v- u is a eccentric vertex of *v* implies *v* is one of the eccentric vertices of *u*. As the distance is geodesic distance the same property is also satisfied by any graph. Consider a subset S, then $d(v,S) = Min[d(u,v): u \in S$. Distance of S, $\sigma(S) = \sum_{v \in V} d(v,S)$. It has been proved by Dankelmann [2] that for a graph G, $avec(G) \leq \frac{\sigma[C(G)]}{n} + r(G)$ and equality holds in the case of trees. The following definitions are taken from [8]. The eccentricity *e(G)* of *G* is a minimum number *k* such that, for each vertex ***v∈V*** with *e(v)≥k,v* is an eccentric vertex of G. Anumberkis called an eccentric number of *G* if, for each vertex *v* with *e(v) = k, v* is an eccentric vertex of *G*. The eccentric spectrum $S_G$of a connected graph *G* is a set of all eccentric numbers in *G*.

IV. Results

Theorem: 1

Let *G: (V, E)* be a connected graph. Then *G* is self centered if and only if ***avec(G) = r(G)***.

Proof:

For any graph G, $avec(G) \leq \frac{\sigma[c(G)]}{n} + r(G)$.

Since G is self-centered we have $\frac{\sigma[c(G)]}{n} = 0$.

Hence we get $avec(G) \leq r(G)$. But for any graph G, r(G) is the minimum eccentricity of all vertices, hence $r(G) \leq avec(G)$. Thus ***avec(G) = r(G)***.

Conversely assume that ***avec(G) = r(G.*** We prove that G is self-centered. $avec(G) \leq \frac{\sigma[c(G)]}{n} + r(G)$and***avec(G)*** = ***r(G)*** gives $0 \leq \frac{\sigma[c(G)]}{n}$.

Case 1

$$\frac{\sigma[C(G)]}{n} = 0$$

$$\sigma[C(G)] = 0 \Rightarrow d[v_1, C(G)] + \cdots + d(v_n, C(G)] = 0$$

Since $d[v_i, C(G)]$ is the distance, it is always greater than or equal to 0. Hence $d[v_i, C(G)] = 0, \forall i = 1,2,..n$.

Thus we see that $v_i \in C(G) \forall i$. Hence G is self centered.

Case 2

Suppose $0 < \frac{\sigma[c(G)]}{n}$. Then $\sigma[C(G)] > 0$. This implies $d[v_1, C(G)] + \cdots + d(v_n, C(G)] > 0$.

Since $d[v_i, C(G)]$ is the distance function, all $d[v_i, C(G)]$ are positive or zero.

Let $d[v_i, C(G)] > 0$ for at least one *i* = *k* , that is $d[v_k, C(G)] > 0$.Since $d[v_k, C(G)] > 0$, v_k is not a member of *C(G)*. So $e(v_k)> r(G)$. Then we get

$$avec(G) = \frac{e(v_1) + e(v_2) + \cdots + e(v_n)}{n} > \frac{1}{n}[n \,.\, r(G)].$$

This gives $avec(G) > r(G)$, which is a contradiction to the assumption that ***avec(G) = r(G)***.

Thus there does not exist a vertex$v_k$such that $d[v_k, C(G)] > 0$. Hence $d[v_k, C(G)] = 0 \forall i$. Therefore $v_i \in C(G) \forall i$.

Hence *G* is self-centered.

Remark

It has been proved by Sameena and Sunitha [6] that a connected fuzzy graph is g - self centered if and only if for every pair of vertices *u* and *v*, *u* is a g - eccentric vertex of *v* implies *v* is one of the g - eccentric vertices of *u*. This result is also true for crisp graphs.

So we can arrive at the following corollary using above remark and Theorem 1.

Corollary: 2

Let *G* be a connected graph. Then *r(G) = avec(G)* if and only if , for every pair of vertices *u*, *v* in *G*, *u* is an eccentric vertex of *v* implies *v* is an eccentric vertex of *u*.

Proposition: 3

Let *G* be any graph with *r(G) = r*.Then $r(G^k) = 1 \ \forall \ k \geq r$.

Proof

Suppose G be any graph with *r(G) = r*. Let $C(G) = v_1, v_2, \ldots, v_m$. Clearly $e(v_1) = e(v_2) = \ldots = e(v_m) = r(G)$ and *r(G) = r*. In G^rwe add an edge between those vertices *u, v* for which *d(u, v) ≤ r*. Consider *G*. Suppose$v_1^* = u_1, v_2^* = u_2, \ldots, v_m^* = u_m$. That is $u_i$is an eccentric vertex of v_i.

Then $d(v_1, u_1) = d(v_2, u_2) = \cdots = d(v_m, u_m) = r$. Thus in G^r there exist arcs $(v_1, u_1), (v_2, u_2) \ldots, (v_m, u_m)$. Also there exist arcs between vertices of *C(G)*and all other vertices in G^ras their distance from *C(G)*is less than *r*.

$\therefore e(v_1) = e(v_2) = \cdots = e(v_m) = 1$.

In G^k where $k > r$, we can still draw the edges $(v_1, u_1), (v_2, u_2) \ldots, (v_m, u_m)$, because *k* is greater than *r*. Hence $e(v_1) = e(v_2) = \cdots = e(v_m) = 1$.

$\therefore r(G^k) = 1 \forall \ k \geq r$.

Proposition: 4

Let G be a graph with ***avec(G)*** = ***r(G)***, then G^r is a complete graph.

Proof

Suppose G be a graph such that ***r(G) = avec(G) = r***. Then from Theorem 1, G is self-centered. So *G = C(G)* and *r(G)= r*, which implies*d(u,v) ≤ r ∀u,v∈ G*. In G^rwe join the vertices *u,v* of *G* if *d(u,v) ≤ r*. Hence we draw an edgebetween each pair of vertices in*G* as *d(u, v) ≤ r*.The resulting graph will be a complete graph, which is G^r.

∴ G^r is complete.

Remark

It is to be noted that the condition ***r(G) = avec(G)*** is necessary for $G^r$to be complete. Below examples, 2 and 3 illustrate this criterion.

Example: 1

The complete bi partite graph $K_{2,3}$has radius ***2*** which equals its average eccentricity. $(K_{2,3})^2$is complete.

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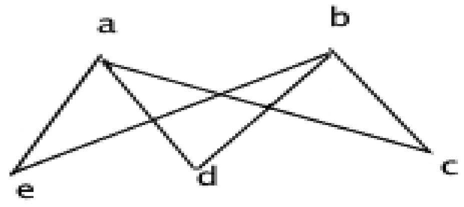
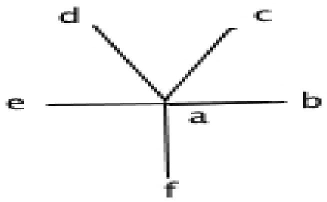


Bi partite Graph

Example: 2

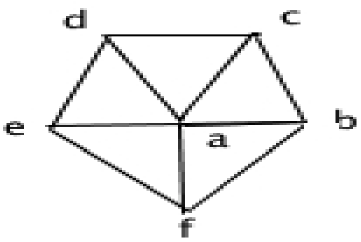
A star graph S_6 has radius 1, which is not equal to its average eccentricity. $(S_6)^1$ is not complete.



Star Graph

Example: 3

A wheel graph W_6 has radius 1, which is not equal to its average eccentricity, $(W_n)^1$ is not complete.



Wheel Garph

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