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The Properties of Beltrami System with Three Characteristic Matrices

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Abstract **– This paper deals with the properties of tensor A and B in elliptic equation of divergence type**

$$\mathrm{div}A(x,\nabla u)=\mathrm{div}B(x,Df)$$

which isderived from the Beltrami system with three characteristic matrices in even dimensions

$$D^t f(x)H(x)Df(x)=J_f^{\frac{2}{n}}(x)G(x)+K(x)D^t f(x)Df(x).$$

Keywords **– Beltrami System With Three Characteristic Matrices, Elliptic System Of Divergence Form, Quasiregu-Larmapping.**

I. Introduction

In complex plane, the study of the property of the solutions of Beltrami systems with one characteristic matrix and two characteristic matrices is very important and we have derived embedded and systematic results. How to generalize the results in two dimensions to high dimensions, the allied sufficient conditions and the regularity of solutions are problems thatare the mathematicians are studying all along (see [1-6]). In this paper, we consider the divergence form elliptic equation

$$\mathrm{div}A(x,\nabla u)=\mathrm{div}B(x,Df), \qquad (1)$$

which is derived from Beltrami system with three characteristic matrices in even dimensions

$$D^t f(x)H(x)Df(x)=J_f^{\frac{2}{n}}(x)G(x)+K(x)D^t f(x)Df(x). \qquad (2)$$

Because divergence form elliptic equation is important for the study of quasiregular mappings, we constructs a bridge of (2) and quasiregular mappings, such that we study the theory of quasiregular mappings with the way of partial differential equations. In (2), $G(x),K(x)\in GL(n)$, $H(x)$ is positive and diagonal matrix, satisfy

(i) $\alpha_1|\xi|^2\le\langle G(x)\xi,\xi\rangle\le\beta_1|\xi|^2,\ 0\le\alpha_1\le\beta_1<\infty;$

(ii) $\alpha_2|\eta|^2\le\langle H(x)\eta,\eta\rangle\le\beta_2|\eta|^2,\ 0\le\alpha_2\le\beta_2<\infty;$

(iii) $\alpha_3|\langle\xi,\ \zeta\rangle|\le\langle K(x)\xi,\zeta\rangle\le\beta_3|\langle\xi,\ \zeta\rangle|,\ \ 0\le\alpha_3\le\beta_3<\infty$.

Our result is based on the following theorem.

Theorem 1.1 Assume that $f\in W^{1,n}_{loc}(\Omega,R^n), n=2k, k=1,2,\cdots$ is a generalized solution of (2) which satisfies (i)(ii)(iii), then $u=f^l\,(l=1,2,\cdots,n)$ are the weak solutions of (1), in which

$$A(x,\nabla u)=\left(\frac{\langle G^{-1}(x)\nabla u,\nabla u\rangle}{H^{ll(x)}}\right)^{\frac{n-2}{2}}G^{-1}(x)\nabla u,$$

$$B(x,Df)=B_1(x,Df)+B_2(x,\nabla u)+B_3(x,\nabla u),$$

$$B_1(x,Df)=(H^{ll}(x)-H^{ll}(x_0))J_f(x)D^{-1}f(x)e^l,$$

$$B_2(x,\nabla u)=H^{ll}(x)J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)\nabla u(x),$$

$$B_3(x,\nabla u)=\sum_{p=1}^{k-1}(-1)^{p+1}C_{k-1}^{p}\left(\frac{\langle G^{-1}(x)\nabla u,\nabla u\rangle}{H^{ll(x)}}\right)^{k-1-p}\cdot\langle G^{-1}(x)K(x)\nabla,\nabla u\rangle^{p}G^{-1}(x)\nabla u.$$

Where H^{ll} denote the lth element of main diagonal line in $H^{-1}(x)$.

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II. The Properties of Operator A

(1) Lipschitz-condition

$$|A(x,h_1)-A(x,h_2)|\le c_1|h_1-h_2|(|h_1|+|h_2|)^{n-2},$$

Proof. Since $G^{-1}(x)\in GL(n)$, there exist orthogonal matrix O_1 and diagonal matrix Γ_1 such that $G^{-1}=O_1\Gamma_1^2O_1^t=P_1^tP_1,\ P_1=(O_1\Gamma_1)^t$. So

$$A(x,h_i)=H^{ll}(x)^{\frac{2-n}{2}}|P_1h_i|^{n-2}P_1^tP_1h_i$$

$$=H^{ll}(x)^{\frac{2-n}{2}}P_1^t|g_i|^{n-2}g_i,\quad g_i=P_1h_i, i=1,2.$$

$$|A(x,h_1)-A(x,h_2)|=|H^{ll}(x)^{\frac{2-n}{2}}P_1^t(|g_1|^{n-2}g_1-|g_2|^{n-2}g_2)|. \qquad (3)$$

In the following, we prove that

$$||g_1|^{n-2}g_1-|g_2|^{n-2}g_2|\le(n-1)|g_1-g_2|(|g_1|+|g_2|)^{n-2}. \qquad (4)$$

In fact, with triangle inequality, we have

$$\begin{aligned}||g_1|^{n-2}g_1-|g_2|^{n-2}g_2|&\le|g_1|^{n-2}|g_1-g_2|\\&\quad+||g_1|^{n-2}-|g_2|^{n-2}|\cdot|g_2|\\&\le(|g_1|+|g_2|)^{n-2}|g_1-g_2|\\&\quad+||g_1|^{n-2}-|g_2|^{n-2}|\cdot|g_2|.\end{aligned} \qquad (5)$$

With the symmetrical characteristic of g_1 and g_2, we have

$$\begin{aligned}||g_1|^{n-2}g_1-|g_2|^{n-2}g_2|&\le(|g_1|+|g_2|)^{n-2}|g_1-g_2|\\&\quad+||g_1|^{n-2}-|g_2|^{n-2}|\cdot|g_1|.\end{aligned} \qquad (6)$$

Let $|g_1|\le|g_2|$, then

$$\begin{aligned}||g_1|^{n-2}-|g_2|^{n-2}|\cdot|g_1|&\le(n-2)|g_2|^{n-3}\cdot||g_1|-|g_2||\cdot|g_1|\\&\le(n-2)|g_2|^{n-3}\cdot|g_1-g_2|(|g_1|+|g_2|)\\&\le(n-2)(|g_1|+|g_2|)^{n-2}|g_1-g_2|.\end{aligned} \qquad (7)$$

From (6) and (7), we have (4).

From (3) and (4), we get

$$|A(x,h_1)-A(x,h_2)|$$

$$\le(n-1)|H^{ll}(x)^{\frac{2-n}{2}}|\cdot|P_1^t|\cdot|g_1-g_2|(|g_1|+|g_2|)^{n-2}. \qquad (8)$$

From $G^{-1}=P_1^tP_1$, and

$\frac{1}{\beta_1}|\xi|^2\le\langle G^{-1}(x)\xi,\xi\rangle\le\frac{1}{\alpha_1}|\xi|^2$, we have

$$\frac{1}{\beta_1}\le|P_1|^2\le\frac{1}{\alpha_1}. \qquad (9)$$

From (8) and (9), we get the Lipschitz-condition.

(2) monotonous inequality

$$< A(x,h_1)-A(x,h_2),h_1-h_2 > \geq c_2|h_1-h_2|^2(|h_1|+|h_2|)^{n-2}.$$

Proof. $< A(x,h_1)-A(x,h_2),h_1-h_2 >$

$$=< H^{ll}(x)^{\frac{2-n}{2}}P_1^t(|g_1|^{n-2}g_1-|g_2|^{n-2}g_2),h_1-h_2 >$$
$$=< H^{ll}(x)^{\frac{2-n}{2}}(|g_1|^{n-2}g_1-|g_2|^{n-2}g_2),g_1-g_2 >$$
$$=H^{ll}(x)^{\frac{2-n}{2}}\{\tfrac{1}{2}[|g_1-g_2|^2(|g_1|^{n-2}+|g_2|^{n-2}) +(|g_1|^2-|g_2|^2)(|g_1|^{n-2}-|g_2|^{n-2})]\}$$
$$\geq H^{ll}(x)^{\frac{2-n}{2}}[\tfrac{1}{2}|g_1-g_2|^2(|g_1|^{n-2}+|g_2|^{n-2})]$$
$$=H^{ll}(x)^{\frac{2-n}{2}}\tfrac{1}{2}< G^{-1}(x)(h_1-h_2),h_1-h_2 >$$
$$\cdot[< G^{-1}(x)h_1,h_1 >^{\frac{2-n}{2}}+< G^{-1}(x)h_2,h_2 >^{\frac{2-n}{2}}]\}$$
$$\geq c|h_1-h_2|^2(|h_1|^{n-2}+|h_2|^{n-2})$$
$$\geq c_2|h_1-h_2|^2(|h_1|+|h_2|)^{n-2}.$$

(3) homogeneity condition

$$A(x,\lambda\xi)=|\lambda|^{n-2}\lambda A(x,\xi),\ \lambda\in R. \qquad (10)$$

Proof. It is easy to get (3.16) from the definition of $A(x,\xi)$.

III. The Property of Operator B in (1)

From (2), we have

$$J_f^{-\frac{2}{n}}(x)H(x)=(D^{-1}f(x))^tG(x)D^{-1}f(x) +J_f^{-\frac{2}{n}}(x)(D^{-1}f(x))^tK(x)D^tf(x).$$

then, for $\forall\xi\in R^n$, we have

$$J_f^{-\frac{2}{n}}(x)< H(x)\xi,\xi >=<(D^{-1}f(x))^tG(x)D^{-1}f(x)\xi,\xi >$$
$$+J_f^{-\frac{2}{n}}(x)<(D^{-1}f(x))^tK(x)D^tf(x)\xi,\xi >$$
$$=< G(x)D^{-1}f(x)\xi,D^{-1}f(x)\xi >$$
$$+J_f^{-\frac{2}{n}}(x)< K(x)D^tf(x)\xi,D^{-1}f(x)\xi >.$$

Then

$$< G(x)D^{-1}f(x)\xi,D^{-1}f(x)\xi >$$
$$=J_f^{-\frac{2}{n}}(x)(< H(x)\xi,\xi >-< K(x)D^tf(x)\xi,D^{-1}f(x)\xi >).$$

Consider (i)(ii)(iii), we have

$$\alpha_1|D^{-1}f(x)\xi|^2\leq< G(x)D^{-1}f(x)\xi,D^{-1}f(x)\xi \leq\beta_1|D^{-1}f(x)\xi|^2.$$
$$\alpha_2|\xi|^2\leq\langle H(x)\xi,\xi\rangle\leq\beta_2|\xi|^2,$$
$$\alpha_3|\xi|^2=\alpha_3|< D^tf(x)\xi,D^{-1}f(x)\xi >|$$
$$\leq< K(x)D^tf(x)\xi,D^{-1}f(x)\xi >=\beta_3|\xi|^2.$$

So

$$|D^{-1}f(x)\xi|^2\leq\tfrac{1}{\alpha_1}J_f^{-\frac{2}{n}}(x)(< H(x)\xi,\xi > -< K(x)D^tf(x)\xi,D^{-1}f(x)\xi >)$$
$$\leq\tfrac{1}{\alpha_1}J_f^{-\frac{2}{n}}(x)(\beta_2-\alpha_3)|\xi|^2.$$

Then

$$|D^{-1}f(x)|^2\leq\tfrac{1}{\alpha_1}(\beta_2+\alpha_3)J_f^{-\frac{2}{n}}(x).$$

$$J_f^{\frac{2}{n}}(x)=\tfrac{< G^{-1}(x)\nabla f^l,\nabla f^l >}{H^{ll}(x)}-< G^{-1}(x)K(x)\nabla f^l,\nabla f^l >$$
$$\leq\tfrac{\beta_2}{\alpha_1}|\nabla f^l|^2-< K(x)\nabla f^l,(G^{-1}(x))^t\nabla f^l >$$
$$\leq\tfrac{\beta_2}{\alpha_1}|\nabla f^l|^2-\alpha_3|< G^{-1}(x)\nabla f^l,\nabla f^l >|$$
$$\leq\tfrac{\beta_2}{\alpha_1}|\nabla f^l|^2-\tfrac{\alpha_3}{\alpha_1}|\nabla f^l|^2$$
$$=\tfrac{\beta_2-\alpha_3}{\alpha_1}|\nabla f^l|^2.$$

Then

$$|J_f(x)D^{-1}f(x)|\leq J_f(x)(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{1}{2}}J_f^{-\frac{1}{n}}(x)$$
$$=(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{1}{2}}J_f^{\frac{n-1}{n}}(x)$$
$$=(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{n}{2}}|\nabla f^l|^{n-1}.$$

So

$$|B_1(x,Df)|=|(H^{ll}(x)-H^{ll}(x_0))J_f(x)D^{-1}f(x)e^l|$$
$$\leq\tfrac{2}{\alpha_2}|J_f(x)D^{-1}f(x)|(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{1}{2}}J_f^{\frac{n-1}{n}}(x)$$
$$\leq\tfrac{2}{\alpha_2}(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{n}{2}}|\nabla f^l|^{n-1}.$$
$$|B_2(x,\nabla f^l)|=|(H^{ll}(x)J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)\nabla f^l|$$
$$\leq\tfrac{1}{\alpha_2}J_f^{\frac{n-2}{n}}(x)|G^{-1}(x)K(x)\nabla f^l|)$$
$$\leq\tfrac{1}{\alpha_2}(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{n-2}{2}}|G^{-1}(x)|\cdot|K(x)|\cdot|\nabla f^l(x)|^{n-1}.$$

Since $K(x)\in GL(n)$, there exist orthogonal matrix O_2 and diagonal matrix Γ_2 such that $K=O_2\Gamma_2^2O_2^t=P_2^tP_2,\ P_2=(O_2\Gamma_2)^t$. Then

$$|< K(x)\xi,\xi >=< P_2^tP\xi,\xi >=|P_2\xi|^2\leq\beta_3|\xi|^2,$$

thus

$$|P_2|^2=\sup_{|\xi|=1}|P_2\xi|^2\leq\beta_3,$$
$$|K(x)|=|P_2^tP_2|\leq|P_2|^2\leq\beta_3.$$

Similarly, we have

$$|G^{-1}(x)|\leq\tfrac{1}{\alpha_1}.$$

So

$$|B_2(x,\nabla f^l)|\leq\tfrac{\beta_3}{\alpha_1\alpha_2}(\tfrac{\beta_2-\alpha_3}{\alpha_1})^{\frac{n-2}{2}}|\nabla f^l(x)|^{n-1}.$$

Taking $n=2k,k=1,2,\cdots$, we have

$$|B_3(x,\nabla f^l)|=|\sum_{p=1}^{k-1}(-1)^{p+1}C_{k-1}^p(\tfrac{\langle G^{-1}(x)\nabla f^l,\nabla f^l\rangle}{H^{ll}(x)})^{k-1-p}$$
$$\cdot< G^{-1}(x)K(x)\nabla f^l,\nabla f^l >^pG^{-1}(x)\nabla f^l|$$
$$\leq\sum_{p=1}^{k-1}C_{k-1}^p|(\tfrac{\langle G^{-1}(x)\nabla f^l,\nabla f^l\rangle}{H^{ll}(x)})^{k-1-p}|$$
$$\cdot|< G^{-1}(x)K(x)\nabla f^l,\nabla f^l >^p||G^{-1}(x)||\nabla f^l|$$
$$\leq\sum_{p=1}^{k-1}C_{k-1}^p(\tfrac{\beta_2}{\alpha_1}|\nabla f^l|^2)^{k-1-p}(\tfrac{\beta_3}{\alpha_1}|\nabla f^l|^2)^p\tfrac{1}{\alpha_1}|\nabla f^l|$$
$$=\sum_{p=1}^{k-1}C_{k-1}^p(\tfrac{1}{\alpha_1})^k\beta_2^{k-1-p}\beta_3^p|\nabla f^l|^{2k-1}$$
$$=(\tfrac{1}{\alpha_1})^k\sum_{p=1}^{k-1}C_{k-1}^p\beta_2^{k-1-p}\beta_3^p|\nabla f^l|^{n-1}.$$

Thus

$$|B(x,Df)|\leq|B_1(x,Df)|+|B_2(x,\nabla u)|+|B_3(x,\nabla u)|$$
$$\leq(c_1+c_2+c_3)|\nabla f^l|^{n-1},$$

in which

$$c_1 = \frac{2}{\alpha_2}\left(\frac{\beta_2-\alpha_3}{\alpha_1}\right)^{\frac{n}{2}},$$

$$c_2 = \frac{\beta_3}{\alpha_1\alpha_2}\left(\frac{\beta_2-\alpha_3}{\alpha_1}\right)^{\frac{n-2}{2}},$$

$$c_3 = \left(\frac{1}{\alpha_1}\right)^k \sum_{p=1}^{k-1} C_{k-1}^{p}\beta_2^{k-1-p}\beta_3^{p}.$$

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