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Beltrami System with Three Characteristic Matrices in Even Dimensions

Chunxia Gao

Abstract **– This paper deals with the Beltrami system with three characteristic matrices in even dimensions**

$$D^t f(x)H(x)Df(x) = J_f^{\frac{2}{n}}(x)G(x) + K(x)D^t f(x)Df(x).$$

Anelliptic equation of divergence type

$$\text{div}A(x,\nabla u) = \text{div}B(x,Df)$$

Isderived under the uniformly elliptic conditionson the matrices $H(x)$, $G(x) \in S(N)$ and $K(x)$ of diagonal and positive.

Keywords **– Beltrami System With Three Characteristic Matrices, Elliptic System of Divergence Form, Quasiregu-Larmapping.**

I. Introduction

Let Ω be a bounded domain in $R^n, n \ge 2$ .Consider a mapping $f = (f^1, f^2, \cdots, f^n) \in W_{loc}^{1,n}(\Omega, R^n)$. Denote the Jacobian matrix and the Jacobian determinant of f by $Df(x) = (\frac{\partial f^i}{\partial x_j})_{1 \le i,j \le n}$ and $J_f(x) = \det Df(x)$ respectively. Denote the transpose and the norm of $Df(x)$ by $D^t f(x)$ and $|Df(x)|$, in which $|Df(x)|^2 = \text{tr}(D^t f(x)Df(x))$. In this paper, we also need another norm of $Df(x)$, denoted by $|Df(x)|_2$, which is defined by

$$|Df(x)|_2 = \sup_{|h| \in S^n} |Df(x)h| \qquad (1)$$

where S^n denotes the unit sphere in R^n . The two matrix norms satisfy

$$|Df(x)|_2 \le |Df(x)| \le n^{\frac{1}{2}} |Df(x)|_2 \cdot \qquad (2)$$

In this paper, we always assume that f is orientation-preserving, i.e. $J_f(x) \ge 0$, a.e. Ω.

Definition 1.1 A mapping f: $\Omega \to R^n$ is called K−quasiregular mapping, $1 \le K < \infty$, if $f(x)$ satisfies

(i) $f \in W_{loc}^{1,n}(\Omega, R^n)$;

(ii) $|Df(x)|^n \le KJ_f(x), a.e.\ x \in \Omega \cdot$

If f is also homeomorphous, then f is called quasiconf-ormal mapping(in [1]).

Beltrami system with one characteristic matrix

$$D^t f(x)Df(x) = J_f^{\frac{2}{n}}(x)G(x). \qquad (3)$$

and Beltrami system with two characteristic matrices

$$D^t f(x)H(x)Df(x) = J_f^{\frac{2}{n}}(x)G(x). \qquad (4)$$

are relevant with quasiregular mappings, in which $G(x), H(x) \in GL(n)$ are $n \times n$ matrices: positive, symmetric with determinant 1, and satisfy the following conditions

(iii) $\alpha_1 |\xi|^2 \le \langle G(x)\xi, \xi \rangle \le \beta_1 |\xi|^2, 0 \le \alpha_1 \le \beta_1 < \infty;$

(iv) $\alpha_2 |\eta|^2 \le \langle H(x)\eta, \eta \rangle \le \beta_2 |\eta|^2, 0 \le \alpha_2 \le \beta_2 < \infty;$

If $f \in W_{loc}^{1,n}(\Omega, R^n)$ is a generalized solution of (4) which satisfies (iii) and (iv), then $f(x)$ is an $(\frac{\beta_1}{\alpha_2})^n$ quasiregular mapping(in [2]).

In complex plane, the study of the property of the solutions of Beltrami systems with one characteristic matrix and two characteristic matrices is very important and we have derived embedded and systematic results. How to generalize the results in two dimensions to high dimensions, the allied sufficient conditions and the regularity of solutions are problems which is the mathematicians are studying all along (see [3-6]). In this paper, we consider the Beltrami system with three characteristic matrices in even dimensions

$$D^t f(x)H(x)Df(x) = J_f^{\frac{2}{n}}(x)G(x) + K(x)D^t f(x)Df(x), \qquad (5)$$

where $G(x), G(x) \in GL(n)$, $H(x)$ is positive and diagonal matrix, satisfy (iii) (iv) and (v)

$\alpha_3 |\langle \xi, \ \zeta \rangle| \le \langle K(x)\xi, \zeta \rangle \le \beta_3 |\langle \xi, \ \zeta \rangle|, \ 0 \le \alpha_3 \le \beta_3 < \infty \cdot$

In the following, we will translate (5) into a divergence form elliptic equation

$$\text{div}A(x,\nabla u) = \text{div}B(x,Df) \cdot \qquad (6)$$

Because divergence form elliptic equation is important for the study of quasiregular mappings, this paper constructs a bridge of (5) and quasiregular mappings, such that we study the theory of quasiregular mappings with the way of partial differential equations.From direct calculation, If $f \in W_{loc}^{1,n}(\Omega, R^n)$ is a generalized solution of (4) , then $f(x)$ is an $\dfrac{n^{\frac{1}{2}} 2^{\frac{n-2}{2}} \beta_1^{\frac{n}{2}}}{\alpha_2^{\frac{n}{2}} - 2^{\frac{n-2}{2}} \beta_3^{\frac{n}{2}}}$ quasiregular mapping.

*Definition 1.2*If for every testingfunction $\phi \in W_0^{1,n}(\Omega, R^n)$ which has compact supports , we have

$$\int_\Omega < A(x,\nabla u), \nabla\phi > dx = \int_\Omega < B(x,Df), \nabla\phi > dx, \qquad (7)$$

then $u = f^l (l = 1,2,\cdots,n)$ are called the weak solutions of divergence form elliptic equation (6).

The main result is the following theorem.

Theorem 1.1 Assume that

$$f \in W_{loc}^{1,n}(\Omega, R^n), n = 2k, k = 1,2,\cdots$$

is a generalized solution of (5) which satisfies (iii)(iv)(v), then $u = f^l (l = 1,2,\cdots,n)$ are the weak solutions of (6), in which

$$A(x,\nabla u) = (\frac{\langle G^{-1}(x)\nabla u, \nabla u \rangle}{H^{ll(x)}})^{\frac{n-2}{2}} G^{-1}(x)\nabla u,$$

$$B(x,Df) = B_1(x,Df) + B_2(x,\nabla u) + B_3(x,\nabla u),$$

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$$B_1(x,Df)=(H^{ll}(x)-H^{ll}(x_0))J_f(x)D^{-1}f(x)e^l,$$

$$B_2(x,\nabla u)=H^{ll}(x)J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)\nabla u(x),$$

$$B_3(x,\nabla u)=\sum_{p=1}^{k-1}(-1)^{p+1}C_{k-1}^p\left(\frac{\langle G^{-1}(x)\nabla u,\nabla u\rangle}{H^{ll}(x)}\right)^{k-1-p}$$

$$\cdot\langle G^{-1}(x)K(x)\nabla,\nabla u\rangle^p G^{-1}(x)\nabla u.$$

Where H^{ll} denote the lth element of main diagonal line in $H^{-1}(x)$.

In the proof of Theorem 1.1, we need the following lemma([6]).

Lemma 1.1 For quasiregular mapping $f\in W_{loc}^{1,n}(\Omega,R^n)$ and arbitrary constant vector $a\in R^n$, in distribution meaning, we have

$$div\{J_f(x)D^{-1}f(x)a\}=0\cdot \tag{8}$$

II. The Proof of Theorem 1.1

Proof. From (5), we have

$$J_f(x)D^{-1}f(x)=J_f^{\frac{n-2}{n}}(x)G^{-1}(x)D^t f(x)H(x)$$
$$-J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)D^t f(x). \tag{9}$$

From (8), for arbitrary constant vector $a\in R^n$, we have

$$\text{div}\{J_f^{\frac{n-2}{n}}(x)G^{-1}(x)D^t f(x)H(x)\cdot a- \tag{10}$$
$$-J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)D^t f(x)\cdot a\}=0.$$

Taking $x_0\in U$, assume $\{e^1,e^2,\cdots,e^n\}$ is aset of standard orthogonal basis in R^n. Let $a=H^{-1}(x_0)e^l, l=1,2,\cdots n$, from (10), we have

$$\text{div}\{J_f^{\frac{n-2}{n}}(x)G^{-1}(x)[\nabla f^l(x)+D^t f(x)(H(x)H^{-1}(x_0)-Id)e^l] \tag{11}$$
$$-J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)D^t f(x)H^{-1}(x_0)e^l\}=0.$$

From (5), we also have

$$J_f^{\frac{n-2}{n}}(x)H^{-1}(x)=Df(x)G^{-1}(x)D^t f(x) \tag{12}$$
$$-Df(x)G^{-1}(x)K(x)D^t f(x)H^{-1}(x).$$

Consider the Lth element of diagonal line in (12), we have

$$<J_f^{\frac{n-2}{n}}(x)H^{-1}(x)e^l,e^l>=<Df(x)G^{-1}(x)D^t f(x)e^l,e^l>$$
$$-<Df(x)G^{-1}(x)K(x)D^t f(x)H^{-1}(x)e^l,e^l>,$$

namely

$$J_f^{\frac{n-2}{n}}(x)H^{ll}(x)=<G^{-1}(x)D^t f(x)e^l,D^t f(x)e^l>$$
$$-H^{ll}(x)<G^{-1}(x)K(x)D^t f(x)e^l,D^t f(x)e^l>$$
$$=<G^{-1}(x)\nabla f^l,\nabla f^l>$$
$$-H^{ll}(x)<G^{-1}(x)K(x)\nabla f^l,\nabla f^l>.$$

From (iv), we have

$$\tfrac{1}{\beta_2}|\eta|^2\le<H^{-1}(x)\eta,\eta>\le\tfrac{1}{\alpha_2}|\eta|^2.$$

Taking $\eta=e^l=(0,\cdots,0,1,\cdots,0)^t$, then

$$\tfrac{1}{\beta_2}\le H^{ll}(x)\le\tfrac{1}{\alpha_2}.$$

So

$$J_f^{\frac{2}{n}}(x)=\frac{<G^{-1}(x)\nabla f^l,\nabla f^l>}{H^{ll}(x)}-<G^{-1}(x)K(x)\nabla f^l,\nabla f^l>. \tag{13}$$

From (9), (11) and (13), we have

$$\text{div}\{\left(\frac{<G^{-1}(x)\nabla f^l,\nabla f^l>}{H^{ll}(x)}-<G^{-1}(x)K(x)\nabla f^l,\nabla f^l>\right)^{\frac{n-2}{n}}G^{-1}(x)\nabla f^l$$
$$+(J_f(x)D^{-1}f(x)+J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)D^t f(x))$$
$$\cdot(H^{-1}(x_0)-H^{-1}(x))e^l-J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)D^t f(x)H^{-1}(x_0)e^l$$
$$=0.$$

Namely,

$$\text{div}\{\left(\frac{<G^{-1}(x)\nabla f^l,\nabla f^l>}{H^{ll}(x)}-<G^{-1}(x)K(x)\nabla f^l,\nabla f^l>\right)^{\frac{n-2}{n}}G^{-1}(x)\nabla f^l$$
$$+J_f(x)D^{-1}f(x)(H^{-1}(x_0)-H^{-1}(x))e^l \tag{14}$$
$$-J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)D^t f(x)H^{-1}(x_0)e^l$$
$$=0.$$

Taking $n=2k, k=1,2,\cdots$, we have

$$\text{div}\left(\frac{<G^{-1}(x)\nabla f^l,\nabla f^l>}{H^{ll}(x)}\right)^{\frac{n-2}{n}}G^{-1}(x)\nabla f^l$$
$$=\text{div}(H^{ll}(x)-H^{ll}(x_0))J_f(x)D^{-1}f(x)e^l$$
$$+\text{div}H^{ll}(x)J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)\nabla f^l$$
$$+\text{div}[\sum_{p=1}^{k-1}(-1)^{p+1}C_{k-1}^p\left(\frac{<G^{-1}(x)\nabla f^l,\nabla f^l>}{H^{ll}(x)}\right)^{k-1-p}$$
$$\cdot<G^{-1}(x)K(x)\nabla f^l,\nabla f^l>^p G^{-1}(x)\nabla f^l].$$

Let $u=f^l$, and

$$A(x,\nabla u)=\left(\frac{\langle G^{-1}(x)\nabla u,\nabla u\rangle}{H^{ll}(x)}\right)^{\frac{n-2}{2}}G^{-1}(x)\nabla u,$$

$$B(x,Df)=B_1(x,Df)+B_2(x,\nabla u)+B_3(x,\nabla u),$$

$$B_1(x,Df)=(H^{ll}(x)-H^{ll}(x_0))J_f(x)D^{-1}f(x)e^l,$$

$$B_2(x,\nabla u)=H^{ll}(x)J_f^{\frac{n-2}{n}}(x)G^{-1}(x)K(x)\nabla u(x),$$

$$B_3(x,\nabla u)=\sum_{p=1}^{k-1}(-1)^{p+1}C_{k-1}^p\left(\frac{\langle G^{-1}(x)\nabla u,\nabla u\rangle}{H^{ll}(x)}\right)^{k-1-p}$$

$$\cdot\langle G^{-1}(x)K(x)\nabla,\nabla u\rangle^p G^{-1}(x)\nabla u.$$

Then Theorem 1.1 is proved.

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Author's Profile

Chunxia Gao
female, major in partial differential equations. College of Electronic and Information Engineering, Hebei University, Baoding, 071002, China.