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The Generalized Solution of Duffing Equation

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***Abstract* – In this paper we describe the result of professor Tyurekhodzhaev's method. The method used in paper allows solving nonlinear differential equation.**

***Keywords* – Delta-function, direct decomposition, Heaviside function, method of extended parameters, nonlinear equations, unity partition.**

I. Introduction

The purpose of this work is to show the advantages of professor Tyurekhodzhaev's method over some asymptotic methods for solving nonlinear differential equations that have a natural geometric interpretation.

Investigation of nonlinear equations is complicated by the fact that they are lack of the principle of superposition, and Green's function apparatus does not work. Taking into account all types of non-linearity in differential equations, it is practically impossible to offer universal research apparatus to study nonlinear problems.

II. Partial Discretization of Differential Equation

In many cases, the autonomous oscillations of conservative systems with one degree of freedom are described by equations of the form (1)

$$\frac{d^2x^*}{dt^{*2}} + f(x^*) = 0 \qquad (1)$$

where f is some nonlinear function of x^*; here an element $\frac{d^2x^*}{dt^{*2}}$ defines system acceleration, and $f(x^*)f(x^*)$ is non-linear restoring force. Let the coordinate $x^* = x_0^*$ define the position of the system equilibrium; then it is clear that $f(x_0^*) = 0$. We also assume that the function f is analytic at $x^* = x_0^*$; then it can be expanded as a Taylor series in a neighborhood of this point.

$$f(x^*) = k_1(x^* - x_0^*) + k_2(x^* - x_0^*)^2 + k_3(x^* - x_0^*)^3 + .. \qquad (2)$$

where $k_n = \frac{1}{n!}\frac{d^n}{dx^{*2}} f(x_0^*)$. Thus equation (1) can also be written as

$$\frac{d^2x^*}{dt^{*2}} + f(x^*) = k_1(x^* - x_0^*) + k_2(x^* - x_0^*)^2 + + k_3(x^* - x_0^*)^3 + \cdots \qquad (3)$$

Equation (3) describes the system motion near the equilibrium position.

Let us rewrite equation (3) in the form

$$\frac{d^2u^*}{dt^{*2}} + k_1u^* + k_2u^{*2} + k_3u^{*3} + \cdots, \qquad (4)$$

where $u^* = x^* - x_0^*$.

Next, we consider a special case of equation (4), which is called the Duffing equation

$$\frac{d^2u^*}{dt^{*2}} + k_1u^* + k_3u^{*3} = 0, \qquad (5)$$

where $k_1 > 0$, and k_3 can be positive as well as negative. Before proceeding to the solution of the original equations we should make the rule to necessarily lead equations to a dimensionless form. To this end we choose some characteristic magnitude of the problem - linear U^* and provisional T^*, and assume

$$u = \frac{u^*}{U^*},\ t = \frac{t^*}{T^*}.$$

Using the rule of differentiating a composite function, we have

$$\frac{d}{dt^*} = \frac{d}{dt}\frac{d}{dt^*} = \frac{1}{T^*}\frac{d}{dt},\ \frac{d^2}{dt^{*2}} = \frac{1}{T^{*2}}\frac{d^2}{dt^2};$$

the equation (5) is transformed as follows

$$\ddot{u} + k_1T^{*2}u + k_3T^{*2}U^{*2}u^3 = 0. \qquad (6)$$

We choose T^* in such a way that $k_1T^{*2} = 1$, and assume $\mu = k_3T^{*2}U^{*2} = k_3\frac{U^{*2}}{k_1}$. Then (6) can be written as

$$\ddot{u} + u + \mu u^3 = 0. \qquad (7)$$

Note that the parameter μ is a dimensionless quantity that characterizes the degree of the system nonlinearity.

Thus the oscillation of the mass body coupled to a nonlinear spring is described by the Duffing equation and initial conditions [1]:

$$\frac{d^2u(t)}{dt^2} + u(t) + \mu u^3(t) = 0,\ u(0) = A,\ u'(0) = 0, \qquad (8)$$

where $u(t)$ is a deviation from the equilibrium position, t – dimensionless time.

We search for approximate solution of the given problem in the form of an asymptotic approach when $\mu \to 0$

$$u(t) = \sum_{n=0}^{\infty} \mu^n u_n(t). \qquad (9)$$

Let substitute equation (9) in the equation with initial conditions, and expand in powers of μ. Then equating the coefficients at equal powers of a small parameter we obtain following expressions to determine u_0 and u_1:

$$u_0'' + u_0 = 0, \quad u_0 = A, \quad u_0' = 0,$$
$$u_1'' + u_1 = -u_0^3, \quad u_1 = 0, \quad u_1' = 0.$$

Problem solution for the leading coefficient of the series has a form

$$u_0(t) = A\cos t. \qquad (10)$$

Substituting this expression into the equation of next series coefficient, and account for identical equality

$$\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t,$$

we get

$$u_1'' + u_1 = -\frac{1}{4}A^3(\cos 3t + 3\cos t), \quad u_1 = 0, \quad u_1' = 0.$$

As a result of integration we find

$$u_1 = -\frac{3}{8}A^3 t\sin t + \frac{1}{32}A^3(\cos 3t - 3\cos t).$$

Hence the solution (direct expansion) of the original problem is given by next formula

$$u(t) = A\cos t + \mu A^3\left[-\frac{3}{8}A^3 t\sin t + + \frac{1}{32}A^3(\cos 3t - 3\cos t) + O(\mu^2)\right] \qquad (11)$$

Based on solution analysis, we conclude

1) We have $\frac{u_1}{u_0} \to \infty$ at $t \to \infty$ because of summand presence $t \sin t$. Therefore obtained solution becomes invalid for large times. It can be used only at $\mu t \ll 1$ (a consequence of applicability condition for $u_0 \gg \mu u_1$ expansion).

2) $t \sin t$ type summands which increases at $t \to \infty$, and limits the range of the applicability of the asymptotic expansions is called secular terms.

In order to obtain a complete picture of the oscillation it is better to eliminate secular terms, this is why we consider parameter-expanding method (Lindstedt-Poincare method) for the equation (8).

Making a replacement of $t = z(1 + \varepsilon\omega_1 + \cdots)$ in (8) we have

$$u''_{zz} + (1 + \varepsilon\omega_1 + \cdots)^2 (u + \mu u^3) = 0. \quad (12)$$

We seek a solution of the form $u = u_0(z) + \mu u_1(z) + + \cdots$. Let substitute it in the equation (12) and equate the coefficients at equal powers of μ, and as a consequence we obtain the following system (primes denote derivatives with respect to z to determine first two terms of the series:

$$u''_0 + u_0 = 0, \quad (13)$$

$$u''_1 + u_1 = -u^3 - 2\omega_1 u_0. \quad (14)$$

The general solution of (13) is given by

$$u_0 = a \cos t\,(z + b), \quad (15)$$

where a and b are constants of integration. Accounting for the equation (15), and after elementary transformations the equation (14) takes the form

$$u''_1 + u_1 = -\frac{1}{4} A^3 \cos\big(3(z + b)\big) - - 2a\left(\frac{3}{8} A^2 + \omega_1\right) \cos(z + b) \quad (16)$$

At $\omega \neq -{}^{3}/_{8} A^2$ a partial solution of the equation (16) consists of a secular term which is proportional to $z \cos(z + b)$. In this case the applicability condition for the expansion ${}^{u_1}/_{u_0} = O(1)$ cannot be satisfied for sufficiently large z. To satisfy this condition it is necessary to assume

$$\omega = -\frac{3}{8} A^2. \quad (17)$$

Then the solution of the equation (16) is defined by

$$u_1 = \frac{1}{32} A^3 \cos\big(3(z + b)\big). \quad (18)$$

Similarly one can define further terms in the expansion. We obtain a solution of the Duffing equation in the next form using expressions (15), (17) and (18)

$$u(t) = A \cos(\omega t + b) + \frac{1}{32} \mu A^3 \cos\big(3(\omega t + b)\big) + O(\mu^2), \quad (19)$$

$$\omega = \left[1 - \frac{3}{8} \mu A^2 + O(\mu^2)\right]^{-1} = 1 + \frac{3}{8} \mu A^2 + O(\mu^2). \quad (20)$$

We apply a method of partial discretization of differential equations (Professor Tyurekhodzhaev's method) to compare obtained results with previously known methods [2]. We start with consideration of the Duffing equation for more detailed explanation of the method.

$$\frac{d^2 u(t)}{dt^2} + u(t) + \mu u^3(t) = 0. \quad (21)$$

Try to solve this equation with the following initial conditions:

$$u(0) = A, \quad u'(0) = 0. \quad (22)$$

Equation (21) is nonlinear; and as we said before investigation of nonlinear equations is complicated due to the absence of the principle of superposition, and nonworking apparatus of Green's function.

Therefore using the theory of "partitions of unity" and taking into account properties of generalized functions we present the equation (20) in the form

$$\ddot{u}(t) + u(t) = -\mu \frac{1}{2} \sum_{k=1}^{n} (t_k + t_{k+1})[u^3(t_k)\delta(t - t_k) - - u^3(t_{k+1})\delta(t - t_{k+1})] \quad (23)$$

Note that we do not set any restriction with respect to the parameter μ (meaning smallness).

Hence the nonlinear Duffing equation (21) is presented as a second order nonhomogeneous differential equation.

Therefore, to find a particular solution we apply a method of variation of parameters

$$u(t) = C_1(t) \cos t + C_2(t) \sin t \quad (24)$$

where $C_1(t)$, $C_2(t)$ are differentiable functions to be determined.

We have

$$\begin{cases} C'_1(t) u_1(t) + C'_1(t) u_2(t) = 0, \\ C'_1(t) u'_1(t) + C'_2(t) u'_2(t) = -\mu \frac{1}{2} \sum_{k=1}^{n} (t_k + t_{k+1}) \times \\ \times [u^3(t_k)\delta(t - t_k) - u^3(t_{k+1})\delta(t - t_{k+1})] \equiv \Phi(t), \end{cases} \quad (25)$$

where $u_1(t) = \cos t$, $u_2(t) = \sin t$ is a fundamental system of solutions of the equation (23).

Solving the system (24) by Cramer's method with respect to $C'_1(t)$, $C'_2(t)$, we obtain

$$\begin{cases} C'_1(t) = -\sin t \cdot \Phi(t), \\ C'_2(t) = \cos t \cdot \Phi(t). \end{cases} \quad (26)$$

After integration of (26)

$$C_1(t) = C_1 + \frac{\mu}{2} \sum_k (t_k + t_{k+1})[\sin t_k\, u^3(t_k) \mathrm{H}(t - t_k) - - \sin t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})],$$

$$C_2(t) = C_2 - \frac{\mu}{2} \sum_k (t_k + t_{k+1})[\cos t_k\, u^3(t_k) \mathrm{H}(t - t_k) - - \cos t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})].$$

Substituting found functions $C_1(t)$, $C_2(t)$ into (24), we obtain general solution of the equation (23):

$$u(t) = C_1 \cos t + C_2 \sin t + \frac{\mu}{2} \Big\{ \sum_k (t_k + t_{k+1})[\sin t_k \times \times u^3(t_k) \mathrm{H}(t - t_k) - \sin t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})\} \times \times \cos t - \frac{\mu}{2} \Big\{ \sum_k (t_k + t_{k+1})[\cos t_k\, u^3(t_k) \mathrm{H}(t - t_k) - - \cos t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})]\} \sin t, \quad (27)$$

Which after differentiation will be

$$\dot{u}(t) = -C_1 \sin t + C_2 \cos t - -\frac{\mu}{2} \Big\{ \sum_k (t_k + t_{k+1})[\sin t_k\, u^3(t_k) \mathrm{H}(t - t_k) -$$

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$$- \sin t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})]\} \sin t -$$

$$-\frac{\mu}{2}\left\{\sum_k (t_k + t_{k+1})[\cos t_k\, u^3(t_k) \mathrm{H}(t - t_k) -\right.$$

$$- \cos t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})]\} \cos t +$$

$$+\frac{\mu}{2}\left\{\sum_k (t_k + t_{k+1})[\sin t_k\, u^3(t_k) \delta(t - t_k) -\right.$$

$$- \cos t_{k+1}\, u^3(t_{k+1}) \delta(t - t_{k+1})]\} \cos t -$$

$$-\frac{\mu}{2}\left\{\sum_k (t_k + t_{k+1})[\cos t_k\, u^3(t_k) \delta(t - t_k)]\right\} \sin t.$$

Using the initial conditions of the problem, we find:

At $t = t_1 = 0$,

$$\begin{cases} C_1 = A, \\ C_2 = 0. \end{cases} \quad (28)$$

Then the solution of $u(t)$ will be written as

$$u(t) = A \cos t + \frac{\mu}{2}\left\{\sum_k (t_k + t_{k+1})[\sin t_k\, u^3(t_k) \times\right.$$

$$\times \mathrm{H}(t - t_k) - \sin t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})]\} \cos t -$$

$$-\frac{\mu}{2}\left\{\sum_k (t_k + t_{k+1})[\cos t_k\, u^3(t_k) \mathrm{H}(t - t_k) -\right.$$

$$- \cos t_{k+1}\, u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})]\} \sin t \quad (29)$$

$$u(t) = A \cos t - \frac{\mu}{2}\sum_k [(t_k + t_{k+1}) \sin(t - t_k) \times$$

$$\times u^3(t_k) \mathrm{H}(t - t_k)] + \frac{\mu}{2}\sum_k [(t_k + t_{k+1}) \sin(t - t_{k+1}) \times$$

$$\times u^3(t_{k+1}) \mathrm{H}(t - t_{k+1})] \quad (30)$$

In accordance with equation (29), $u(t)$ functions at points tk will be expressed as

$u(t_1) = A$,

$u(t_2) = A \cos t_2$,

$u(t_3) = A \cos t_3 - \frac{\mu}{2} t_3 \sin t\, (t_3 - t_2) u^3(t_2)$,

$u(t_4) = A \cos t_4 - \frac{\mu}{2}(t_3 - t_1) \sin(t_4 - t_2)\, u^3(t_2) -$

$- \frac{\mu}{2}(t_4 - t_2) + \sin(t_4 - t_3)\, u^3(t_3)$,

....

$u(t_k) = A \cos t_k - \frac{\mu}{2}(t_{k-1} - t_{k-3}) \sin(t_k - t_{k-2}) \times$

$\times u^3(t_{k-2}) - \frac{\mu}{2}(t_k - t_{k-2}) \sin(t_k - t_{k-1})\, u^3(t_{k-1})$.

Next, using the method of mathematical induction, we construct an expression of the required function at an arbitrary point t_k.

$$u(t_k) = A \cos t_k - \frac{\mu}{2}\sum_{j=1}^{k-2} (t_{j+2} - t_j) \times$$

$$\times \sin(t_k - t_{j+1})\, u^3(t_{j+1}) \quad (31)$$

for equally spaced argument values $t_k = t_1 + \Delta t(k - 1)$, $t_{j+2} - t_j = 2\Delta t$

$$u(t_k) = A \cos t_k - \mu \Delta t \sum_{j=1}^{k-2} \sin(t_k - t_{j+1})\, u^3(t_{j+1}). \quad (32)$$

Results are shown on Fig. 1-3 for different oscillating amplitudes and the degree of the μ system nonlinearity.

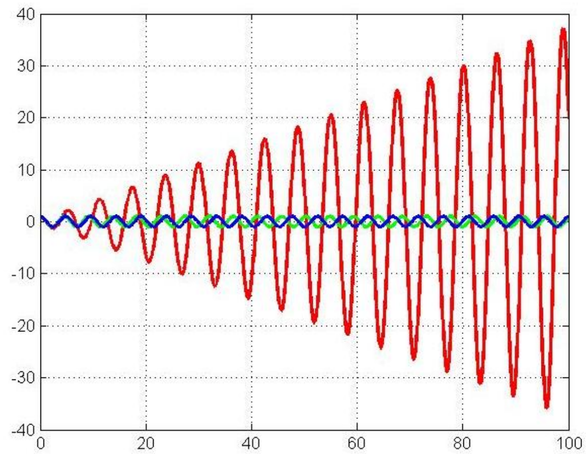

Fig.1. A plot for amplitude equal to 1 and $\mu = 1$.

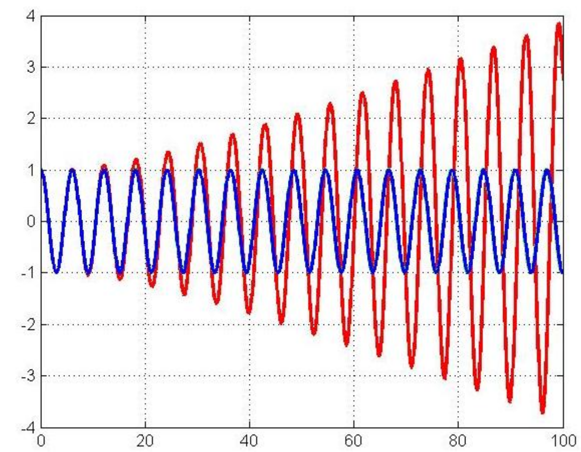

Fig. 2. A plot for amplitude equal to 1 and$\mu = 0.1$.

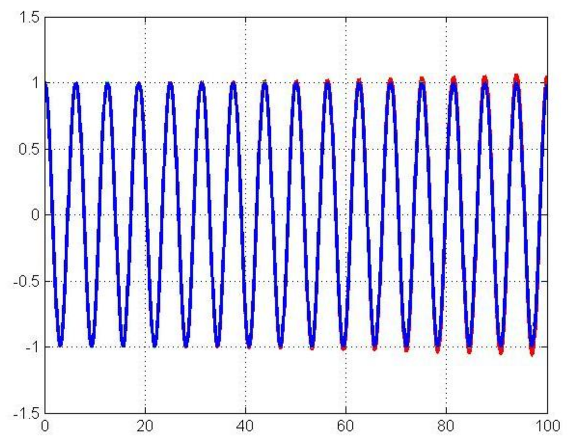

Fig.3. A plot for amplitude equal to 1 and $\mu = 0.01$.

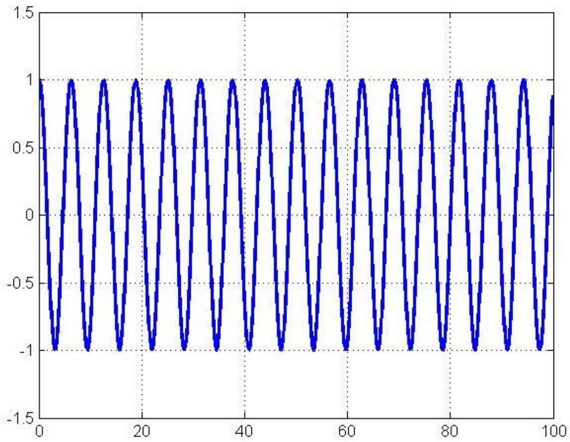

Fig.4. A plot for amplitude equal to 1 and$\mu = 0.001$.

III. Conclusion

Figures comparison allows the following conclusions:

Oscillation period does not depend on the amplitude in case of not large oscillation. Secular term $t \sin t$ is present in direct expansion, and obtained solution becomes invalid for large times. Lindstedt-Poincare method gives the opportunity to obtain uniformly valid solution.

Professor Tyurekhodzhaev's method is the most effective out of three applied methods because it gives the most accurate approximate solution at small and large times.

Because of oscillations nonlinearity period of oscillation depends on the amplitude$T = T(A)$.

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Author's Profile

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was born in Karatau, Kazakhstan, in 1955. He graduated from Department of Mechanics and Applied Mathematics of the Kazakh National University with a degree in Mechanics, in 1977. He received Ph.D. in Physics and Mathematics from the same university in 1999.

In 1977, he joined Department of Mining of the Kazakh National Technical University (KazNTU) as a Lecturer and in 1979 became a Reader in Department of Higher Mathematics. In 2015 he became head of the research work on the topic "The development of a new methodology for advanced horizontal and deviated well completion design" at the Department of Mathematics of the KazNTU.

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