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A Lipschitz Characterization of Commutators for Singular Integrals

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Abstract **– In this paper, we give a Lipschitz characterization of commutators formed by oscillatory integral operators and Lipschitz functions. Furthermore, we also obtain a weighted estimates of these commutators on Lebesgue sapce.**

Keywords **– Oscillatory Integral, C-Z Kernel, Lipschitz Function.**

I. Introduction

Oscillatory integrals in one form or another have been an essential part of harmonic analysis from the very beginnings of that subject. Many operators in harmonic analysis or partial dierential equations are related to some versions of oscillatory integrals, such as the Fourier transform, the Bochner-Riesz means, and the Radon transform which has important applications in the CT technology. The theory of commutators is an important extension of that of oscillatory integral([4,5]). This paper is devoted to study a class of oscillatory integrals defined by Ricci and Stein[1]

$$Tf(x)=p\cdot v\int_{\mathbf{R}^n} e^{ip(x,y)}K(x-y)f(y)dy,$$

where $p(x,y)$ is a real-valued polynomial defined on $\mathbf{R}^n\times\mathbf{R}^n$ and the function K is a Calderon-Zygmund kernel, i.e. $K\in C^\infty(S^{n-1})$ satisfies

$$|K(x)|\le\frac{\Omega(x)}{|x|^n},\ \int_{S^{n-1}}\Omega(x)dx=0。$$

Some boundedness theory of T, see [1-3] for example.

Let b be locally integral function on $\mathbf{R}^n$. Then the commattaor fromed by T and b can be defined as $T_bf(x)=\int_{\mathbf{R}^n} e^{ip(x,y)}(b(x)-b(y))K(x-y)f(y)dy$.

Many works about the above operator are focused on the case when b are BMO functions. Inspired by [6], we study the weighted boundedness of $T_bf(x)$ when b is Lipschitz function. Using Riesz transform, we also obtain a Lipschitz characterization of $T_bf(x)$.

II. Preliminaries

Let $Q(x_0,r)$ be a cube centeded in x_0 and with ridus r. Then the average of f in Q is

$$f_Q=\frac{1}{|Q|}\int_Q f(x)dx,$$

the mean oscillatory of f in Q is

$$\Omega(f,Q)=\frac{1}{|Q|}\int_Q|f-f_Q|dx。$$

Let φ be an increasing positive function and

$$\mathrm{BMO}_\varphi(\mathbf{R}^n):=\{f:\Omega(f,Q)\le C\varphi(r)\}.$$

It is obvious that $\mathrm{BMO}_1(\mathbf{R}^n)=\mathrm{BMO}(\mathbf{R}^n)$ when $\varphi\equiv1$.

Lemma 2.1 [7] For any $a\in\mathbf{R}$, we have $\Omega(f,Q)\le\frac{2}{|Q|}\int_Q|f(x)-a|dx$.

If $0<\alpha\le1$, then the Lipschitz function space can be defined by

$$\mathrm{Lip}_\alpha:=\left\{f:|f(x)-f(y)|\le C|x-y|^\alpha\right\}。$$

It is meaningful to study the problem of commutators fromed by operators and Lipschitz functions, more results about this topic, see e.g. [8-9].

Lemma 2.2[6] If $\varphi(r)=r^\alpha$, then $\mathrm{BMO}_\varphi=\mathrm{Lip}_\alpha$,.

Definition 2.1[10] If $0<\alpha<n$, th Riesz potential of f can be defined by

$$I_\alpha(f)(x)=r(\alpha)\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,$$

where $r(\alpha)=\pi^{\frac{n}{2}}2^\alpha\Gamma\left(\frac{\alpha}{2}\right)\Big/\Gamma\left(\frac{n-\alpha}{2}\right)$.

Lemma 2.3 [10] If $0<\alpha<n, 1<p<\frac{n}{\alpha}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, $w(x)\in A(p,q)$, then

$$\|I_\alpha f\|_{L^q(w)}\le C\|f\|_{L^p(w)}。$$

In 1992, Sawyer and Wheeden proved the following weak result as follows

Lemma 2.4[11] Assumme that $w(x),\upsilon(x)\in L_{loc}^1(R^n)$ $w(x),\upsilon(x)\ge0$, and (w,υ) satisfies the Fefferman-Phong condition:

$$|Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\left(\frac{1}{|Q|}\int_Q w(x)^{qr}dx\right)^{\frac{1}{qr}}\left(\frac{1}{|Q|}\int_Q w(x)^{-p'r}dx\right)^{\frac{1}{p'r}}\le C.$$

Here $1\le r<\infty$, $1<p\le q<\infty$, then

$$\left(\int_{\mathbf{R}^n}|I_\alpha f(x)w(x)|^q dx\right)^{\frac{1}{q}}\le C\left(\int_{\mathbf{R}^n}|f(x)\upsilon(x)|^p dx\right)^{\frac{1}{p}}.$$

It is easy to check that when $r=1$, $\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}=0$, $\upsilon(x)=w(x)$, Lemma 2.4 is Lemma 2.3.

III. Main Results

We begin with a Lipschitz characterization of T_b as follows

Theorem 3.1 Let $f\in L^p(w)$, $\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}=0$, $p(x,y)=p_1(x)+p_2(y)$. Then if $w\in A(p,q)$, we have $b\in Lip_\alpha$ if and only if there is a constant $C>0$, such that

$$\|T_b f\|_{L^q(w)}\le C\|f\|_{L^p(w)}. \tag{3.1}$$

Proof: Assume that (3.1) is true, we show $b\in Lip_\alpha$. Take $z_0\ne 0,\delta>0$ and write

$\frac{1}{K(z)}=\sum a_n e^{i\vec{v}_n\cdot z}$. Let $z_1=\frac{z_0}{\delta}$, if $|z-z_1|<\sqrt{n}$, we have

$$\frac{1}{K(z)}=\frac{\delta^{-n}}{K(\delta z)}=\delta^{-n}\sum a_n e^{i\vec{v}_n\cdot\delta z}. \tag{3.2}$$

For fixed cube $Q=Q(x_0,r)$, $y_0=x_0-rz_1$, $Q'=Q(y_0,r)$, If $x\in Q, y\in Q'$, then

$$\left|\frac{x-y}{r}-z_1\right|\le\left|\frac{x-x_0}{r}\right|+\left|\frac{y-y_0}{r}\right|\le\sqrt{n}.$$

Set $S(x)=\operatorname{sgn} f(x)-f_{Q'}$, then we have

$$\int_Q|f(x)-f_{Q'}|dx=\int_Q(f(x)-f_{Q'})S(x)dx$$
$$=\frac{1}{|Q'|}\int_Q\int_{Q'}(f(x)-f(y))S(x)dydx.$$

By (3.2),

$$\int_Q|f(x)-f_{Q'}|dx=r^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}e^{ip(x,y)}(f(x)-f(y))\frac{r^nK(x-y)}{K(\frac{x-y}{r})}\times S(x)\chi_Q(x)\chi_{Q'}(y)\frac{1}{e^{ip(x,y)}}dydx$$

Choosing $f_n(y)=e^{-i(\frac{\delta}{r}v_n\cdot y-p_2(y))}\chi_{Q'}(y)$,

$g_n(x)=e^{i(\frac{\delta}{r}v_n\cdot x-p_1(y))}S(x)\chi_Q(y)$, (3.1) gives

$$\int_Q|f(x)-f_{Q'}|dx$$
$$=C\sum a_n\int\int e^{ip(x,y)}(f(x)-f(y))K(x-y)f_n(y)g_n(x)dydx$$
$$=C\sum a_n\int T_b f_n(x)g_n(x)dx$$
$$\le C\sum a_n\int_Q|T_b f_n(x)|dx.$$

Since $f_n\in L^p$, $\|f_n\|_{L^p}=|Q'|^{\frac{1}{p}}=|Q|^{\frac{1}{p}}=r^{\frac{n}{p}}$,

$\|T_b f_n\|_{L^q}\le C\|f_n\|_{L^p}=Cr^{\frac{n}{p}}$. Moreover

$$\int_Q|T_b f_n|\le C\|T_b f_n\|_{L^q}|Q|^{1-\frac{1}{q}}\le Cr^{n(\frac{1}{p}-\frac{1}{q})}|Q|.$$

And hence

$$\int_Q|f(x)-f_{Q'}|dx\le C\sum a_n r^{n(\frac{1}{p}-\frac{1}{q})}|Q|=C|Q|r^{n(\frac{1}{p}-\frac{1}{q})}.$$

If we take $\varphi(r)=r^\alpha$, Lemma 2.1shows that $\Omega(f,Q)\le C\varphi(r)$, and finally $b\in Lip_\alpha$ by Lemma 2.2.

To prove the converse, we first note that

$$|T_b f(x)|=\left|\int e^{ip(x,y)}(b(x)-b(y))K(x-y)f(y)dy\right|$$
$$\le\int|(b(x)-b(y))K(x-y)f(y)|dy$$
$$\le C\|b\|_{Lip_\alpha}\int\frac{1}{|x-y|^{n-\alpha}}|f(y)|dy$$
$$=CI_\alpha(|f|)(x).$$

Using Lemma 2.3, one has

$$\|T_b f(x)\|_{L^q(w)}=\left(\int|T_b fw(x)|^q dx\right)^{\frac{1}{q}}$$
$$\le C\left(\int|I_\alpha(|f|)(x)w(x)|^q dx\right)^{\frac{1}{q}}$$
$$\le C\|f\|_{L^p(w)}.$$

We have complted the proof.

Corollary 3.1 If (w,v) satisfies the Fefferman-Phong condition and $b\in Lip_\alpha$, then there is a constant $C>0$ such that

$$\|T_b f\|_{L^q(w)}\le C\|f\|_{L^p(v)}.$$

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