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Weighted Norm Inequalities for Singular Integrals With Oscillating Kernels

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Abstract: **This paper deals with the boundedness of singular integral operators with oscillating kernels on weighted Morrey space, including the weak type and strong type weighted norm inequalities. Furthermore, the corresponding boundedness for their commutators are also considered.**

Keywords: **Oscillating Kernel, Weighted Morrey Space, A_pWeights.**

I. Introduction

To investigate the local behavior of solutions to second order elliptic partial differential equations, Morrey [1] first introduced the classical Morrey space $M_{p,q}(R^n)$ with the norm

$$\|f\|_{M_{p,q}(R^n)}=sup_{B\subset R^n}\left(\frac{1}{|B|^{1-\frac{p}{q}}}\int_B |f(x)|^p dx\right)^{\frac{1}{p}},$$

where $f\in L^p(R^n)$ and $1\le p\le q<\propto$. Here and after, B denotes a ball in R^n.$M_{p,q}(R^n)$was an expansion of $L^p(R^n)$ in the sense that $M_{p,p}(R^n)=L^p(R^n)$. Since then, many works associated with boundedness of operators on Morrey spaces are reported. In [2], Chiarenza and Frasca obtained the boundedness of two classical operators in harmonic analysis on $M_{p,q}(R^n)$, i.e, the Hardy-Littlewood maximal function

$$\mathrm{Mf(x)}=sup_{x\in B}\frac{1}{|B|}\int_B |f(y)|dy$$

and the Calder´on-Zygmund singular integral operator

$$\mathrm{T'f(x)}=p.v.\frac{1}{|B|}\int_{R^n} K(x-y)f(y)dy \quad ,$$

where K is a Calder´on-Zygmund kernel. More works about the boundedness of operators on Morrey type spaces, see for example [3, 4, 5, 6, 7]. In [8], Komori and Shirai introduced the weighted Morrey space, which is a natural generalization of weighted Lebesgue space. Let $1\le p<\infty, 0<k<1$ and wbe a weight function. Then the weighted Morrey space$M_{p,k}(w)$ was defined by

$$M_{p,k}(w)=\left\{f\in L^p_{loc}(w): \|f\|_{M_{p,k}(w)}<\infty\right\},$$

Where

$$\|f\|_{M_{p,k}(w)}=sup_B\left(\frac{1}{(w(B))^k}\int_B |f(x)|^p\, w(x)dx\right)^{\frac{1}{p}}.$$

It is obvious that if $\mathrm{w}=1, \mathrm{k}=1-\frac{p}{q}$, then $M_{p,k}(w)=M_{p,q}(R^N)$.Komori and Shirai obtained the boundedness of M, T'on $M_{p,k}(w)$ with $1\le$ p<∞ and w ∈Ap. Here Ap denotes the Muckenhoupt classes [9].

Given a positive real number $\alpha>0$, $\alpha\neq 1$, the oscillating kernel was define by

$$K_\alpha(x)=e^{i|x|^\alpha}(1+|x|)^{-n}.$$

The singular integral operator with oscillating kernel can be denoted by

$$\mathrm{T}f(x)=\mathrm{K}_\alpha * f(x).$$

It is well known that the operator Twas bounded on $L^p(R^n)(1<p<\infty)$ (cf. [10, 11]). Chanillo, Kurtz and Sampson obtained the weighted weak type (1,1) and weighted strong type (p,p) inequalities for T on Lebesgue spaces ([12, 13]).

This paper is devoted to the weighted bondedness of T on $M_{p,k}(w)$ following from some ideas which were developed in [14] dealing with Lebesgue spaces.

The main results of this paper can now be formulated as

Theorem 1.1. Let $w\in A_1$ and $0<k<1$. Then there exists a constant $C>0$such that

$$sup_{\lambda>0}\lambda w(\{x\in B: |Tf(x)|>\lambda\})\le\|f\|_{M_{p,k}(w)}w(B)^k.$$

Theorem 1.2. Let $1<p<\infty$, $0<k<1$and $w\in A_p$. Then there exists a constant $C>0$ such that

$$\|Tf\|_{M_{p,k}(w)}\le C\|f\|_{M_{p,k}(w)}.$$

Theorem 1.1 and Theorem 1.2 will be proved in Section 2. Section 3 presents the weighted norm inequalities for the corresponding commutators of T. Throughout this paper all definitions and notations are standard. C denote a positive constant which may vary from line to line but will remain independent of the relevant quantities. $\mathrm{B}=\mathrm{B}(x_0,\mathrm{r})$ denotes the ball with center $\mathrm{x}_0$and radius r. Given $\lambda>0$, $\lambda\mathrm{B}=\mathrm{B}(x_0,\lambda\mathrm{r})$.

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II. Weighted Estimates for Oscillating Kernels

We begin this section with some properties of $A_p$weight classes which play important role in the proofs of our main results.

Lemma 2.1. [15] Let $1 \le p < \infty$ *and* $w \in A_p$. *Then the following statements are true*

(1) There exists a constant Csuch that

(2.1) w (2B) ≤ Cw (B).

(2) There exists a constant C>1 such that

(2.2) w (2B) ≥ Cw (B).

(3) There exist two constants C and r >1 such that the following reverse H¨older inequality holds for every ball B⊂R^n

(2.3)$\left(\frac{1}{|B|}\int_B |w(x)|^r\,dx\right)^{\frac{1}{r}} \le C\frac{1}{|B|}\int_B |w(x)|\quad dx$

Our argument based heavily on the following well-known results about Ton weighted Lebesgue spaces

Lemma 2.2. [13] Letα >0, α ≠ 1.

(1) If $w \in A_1$, then there is a constant C>0 such that

(2.4) $sup_{\lambda>0}\lambda w(\{x\in B: |Tf(x)|>\lambda\}) \le \|f\|_{L^1(w)}$.

(2) If $w \in A_p(1<p<\infty)$, then there is a constant C>0 such that

(2.5) $\|Tf\|_{L^p(w)} \le C\|f\|_{L^p(w)}$.

Proof of Theorem 1.1.

Decompose $f = f_{\chi 2B} + f_{\chi(2B)^c} =: f_1 + f_2$. For any given $\lambda > 0$, we write

$$w(\{x\in B: |Tf(x)|>\lambda\}) \le w\left(\left\{x\in B: |Tf_1(x)|>\frac{\lambda}{2}\right\}\right) + w\left(\left\{x\in B: |Tf_2(x)|>\frac{\lambda}{2}\right\}\right) := I + II$$

An application of (2.1) and (2.4) yields that

(2.6)$I \le \frac{C}{\lambda}\|f\|_{M_{1,k}(w)} w(B)^k$

Next we turn to deal with the term II. An elementary estimate shows

$$w\left(\left\{x\in B: |Tf_2(x)|>\frac{\lambda}{2}\right\}\right) \le \frac{C}{\lambda}\int_{\left\{x\in B: |Tf_2(x)|>\frac{\lambda}{2}\right\}} |Tf_2(x)|\quad w(x)dx.$$

We note that

(2.7) $|Tf_2(x)| \le C\sum_{j=1}^{\infty}\frac{1}{|2^jB|}\int_{2^jB}|f\ \ (y)|\quad dy$ Substituting (2.7) into IIgives

(2.8)$II \le \frac{C}{\lambda}\|f\|_{M_{1,k}(w)} w(B)^k$.

Therefore, Theorem 1.1 is a by-product of (2.6) and (2.8).

Proof of Theorem 1.2.

It suffices to show that

(2.9) $\frac{1}{w(B)^k}\int_B |Tf(x)|^p\, w(x)dx \le C\|f\|_{M_{p,k}(w)}$.

Let $f = f_{\chi 2B} + f_{\chi(2B)^c} =: f_1 + f_2$ and hence $Tf\ \ (x) = Tf_1(x) + Tf_2(x)$. Since T is a linear operator, one has

(2.10) $$\frac{1}{w(B)^k}\int_B |Tf(x)|^p\, w(x)dx \le \frac{1}{w(B)^k}\int_B |Tf_1(x)|^p\, w(x)dx + \frac{1}{w(B)^k}\int_B |Tf_2(x)|^p\, w(x)dx := J + JJ$$

We are now in a position to estimate the term JJ. After noticing (2.7), H¨older's inequality and the Ap condition imply that

$$\int_{2^jB} |f\ \ (y)|\quad dy \le C\|f\|_{M_{p,k}(w)} \left|2^{j+1}B\right| w(2^{j+1}B)^{\frac{1}{p}(k-1)}$$

Then, using (2.2) we obtain that

(2.11) $J \le C\|f\|_{M_{p,k}(w)}$.

We get (2.9) by (2.10)and (2.11). The proof of Theorem 1.2 is completed.

III. Weighted Estimates for Commutators

A locally integrable function bis said to be in BMO(R^n) if for any ball B ⊂R^n

$$\|b\|_{BMO(R^n)} = sup_B \frac{1}{|B|}\int_B |b(x) - b_B|dx < \infty,$$

where $b_B = \frac{1}{|B|}\int_B |b(x)|dx$. For a locally integrable function b, the commutator formed by T and b are defined by

$$Tb := [b, T]f(x) = b(x)Tf(x) - T(bf)(x).$$

The aim of this section is to set up the weighted boundedness for the commutators formed by T and BMO(R^n) functions. We first formulate some remarks about BMO(R^n) functions.

Lemma 3.1. [16], [17] Let $1 \le p < \infty$ *and* $b \in BMO(R^n)$. *Then for any ball* $B\subset R^n$, *the following statements are true*

(1) There exist constants C1, C2 such that for all α>0

$$|\{x\in B: |b(x) - b_B| > \alpha\}| \le C_1|B|e^{-C_2\alpha/\|b\|_{BMO(R^n)}}$$

The inequality is also called John-Nirenberg inequality.

(2) $|b_{2^\lambda B} - b_B| \le 2^n\lambda\|b\|_{BMO(R^n)}$

Lemma 3.2. [18] Let $w \in A_\infty$. Then the following statements are equivalent

(1) $\|b\|_{BMO(R^n)} \approx sup_B \left(\frac{1}{|B|}\int_B |b(x)-b_B|^p\, dx\right)^{\frac{1}{p}}$;

(2) $\|b\|_{BMO(R^n)} \approx sup_B inf_a \frac{1}{|B|}\int_B |b(x)-a| \quad dx$;

(3) $\|b\|_{BMO(w)} \approx sup_B \frac{1}{w(B)}\int_B |b(x)-b_{B,w}|w(x) \quad dx$.

Our main result of this section is

Theorem 3.3. Let p,k,w be the same as that of Theorem 1.2 and $b \in BMO(R^n)$. Then there exists a constant C>0 such that

$$\|T_b f\|_{M_{p,k}(w)} \le C\|f\|_{M_{p,k}(w)} .$$

To prove Theorem 3.3, we need the following Lemmas.

Lemma 3.4.[19] Suppose that $w \in Ap(1<p<\infty)$. Then for any $b \in BMO(R^n)$, there exists constants C>0 such that

$$\|T_b f\|_{L^p(w)} \le C\|f\|_{L^p(w)}.$$

Lemma 3.5. [5] Let $B = B(x0,r), 0<k<1$, $b \in BMO(R^n)$ and $1<p<\infty$. Then the inequality

$$\left(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}|b(y)-b_{B,w}|dy\right)^p w(B)^{1-k} \le C\|f\|^p_{M_{p,k}(w)}$$

holds for every $y \in (2B)^C$, where $(2B)^C = R^n/2B$.

Proof of Theorem 3.3. The task is now to find a constant Csuch that for fixed ball $B = B(x0,r)$, we can obtain

$$(3.1)\ \frac{1}{w(B)^k}\int_B |T_b f(x)|^p\, w(x)dx \le C\|f\|_{M_{p,k}(w)}.$$

We decompose $f = f_{\chi 2B} + f_{\chi(2B)^C} =: f_1 + f_2$ and consider the corresponding splitting

$$\int_B |T_b f(x)|^p\, w(x)dx \le \int_B |T_b f_1(x)|^p\, w(x)dx + \int_B |T_b f_2(x)|^p\, w(x)dx := L + LL$$

An application of Lemma 3.4 yields that

$$(3.2)\ L \le C\|f\|_{M_{p,k}(w)} w(B)^k$$

To estimate the term LL, we first note that

$$LL \le C\left(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}dy\right)^p \int_B |b(x)-b_{B,w}|^p w(x) \quad dx + C\left(\int_{|x_0-y|>2r}\frac{|f(y)|}{|x_0-y|^n}|b(y)-b_{B,w}|dy\right)^p w(B) \quad := LL_1 + LL_2$$

By Lemma 3.5, one has

$$LL_2 \le C\|f\|_{M_{p,k}(w)} w(B)^k.$$

To get the desired estimate, we are led to estimate the term LL_1. This estimate will be done via (2.1), (2.3) and Lemma 3.2. In fact

$$LL_1 \le C\left(\sum_{j=1}^{\infty}\frac{1}{|2^j B|}\int_{2^{j+1}B}\frac{|f(y)|}{|x_0-y|^n}dy\right)^p \int_B |b(x)-b_{B,w}|^p w(x) \quad dx$$

$$\le C\|f\|^p_{M_{p,k}(w)}\left(\sum_{j=1}^{\infty}\left(\frac{w(B)^{\frac{1-k}{p}}}{w(2^{j+1}B)^{\frac{1-k}{p}}}\right)^p\right) \quad w(B)^k \le C\|f\|_{M_{p,k}(w)} w(B)^k$$

Hence

$$(3.3)\ LL \le C\|f\|_{M_{p,k}(w)} w(B)^k \quad .$$

Combing (3.2) with (3.3), we obtain (3.1), which is the desired conclusion.

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