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Mathematical Modeling for the Estimation of Probability of There Being n Bacteria at Time t in Two Parameters

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Abstract: **In this paper, mathematical modeling is done for the estimation of probability of there being n bacteria at time t using generating function with certain assumptions. Formula is derived to know the probability of there being n bacteria at time t in terms of two parameters α and β. These probabilities are computed for various values of t by assigning certain values to the parameters. These calculations are shown graphically.**

Keywords: **Mathematical Modeling, Probability of N Bacteria.**

I. Introduction

If number of bacteria present in culture is known then one can calculate the amount of protein or DNA. Microbial enumeration is useful in the areas of public health. Microbiologists test food, milk and water for the number of microbial pathogens present to know whether these are safe for human consumption or not. Thus, the counting of bacteria is useful in pharmacy and biochemical studies. The growth of bacteria has been discussed by Delbruck [1], Luria[2] and Grover[3]. For theoretical estimation one can use the mathematical modeling. Mathematical model describes a system in terms of a set of variables and a set of equations either in algebraic or differential equations, or inequalities. The system may be biological or physical or social system. Mathematical modeling is applied in the problems of diverse disciplines like biological sciences, medicine, genetics, physiology, pharmacokinetics, bio-economics and, business and administration. These methods provide a frame work for interpreting and integrating the data. The development of any mathematical model has two aims, understanding and prediction. Some cases, it may require to predict the amount of bacteria present in a time length. In recent years, many researchers drew their attention to the problems in mathematical models in several interdisciplinary sciences. Mathematical modeling was initiated in 1925 by Lotka[4]. Kapur J.N.[5, 6] examined various mathematical models in biological and mathematical sciences.

In this paper, using the mathematical modeling a partial differential equation is obtained for the problem under consideration where the probability of there being n bacteria as dependent variable and, the time t and number of bacteria n as independent variables. The partial differential equation is solved using the generating function with the assumption that there are r number of bacteria at time t=0. The various probabilities are computed for various values of time t and they are shown graphically. From the graph we can find the time at which the probability is maximal and it can be used to assess whether there will be n bacteria or not.

Notations:
P(n,t) – Probability of there being n bacteria at time t
C(m,n) – ${}^{m}C_{n}$, where m and n are non-negative integers
$p_m^b(t)\ \delta t$ – probability that a cell divided into two cells in time $(t, t+\delta t)$ when initial number of bacteria is m
$p_m^d(t)\ \delta t$ – probability that a cell die in time $(t, t+\delta t)$ when initial number of bacteria is m
α, β – parameters.

II. Methods

Formation and solution of the problem:

At time t we assume that there are m bacteria in a fluid contained in a sterile test tube. The value m depends on time t. The birth rate and death rate depends on the fluid contained in the test tube. For example, the birth rate and death rates of streptococcus lactis in milk are different from the medium lactose broth[8]. These are also depends on the temperature [7]. Thus the birth and death rates are different for different set of conditions like temperature and pressure. We also assume that the probability of a bacteria reproducing in time $(t, t+\delta t)$ is $p_m^b(t)\ \delta t$, the probability of a bacteria dying in time $(t, t+\delta t)$ is $p_m^d(t)\ \delta t$, the probability of more than one birth or death occur in time $(t, t+\delta t)$ is zero, $p_m^b(t)$ and $p_m^d(t)$ are proportional to m,i.e., $p_m^b(t)=m\alpha$ and $p_m^d(t)=m\beta$ where α and β are constants. The probability of the number of bacteria remaining constant in time $(t, t+\delta t)$ is $1-p_m^b(t)\delta t-p_m^d(t)\ \delta t$. The constants α and β varies from one bacteria to the other when the fluid contained in tube changes. Now applying the law of compound and total probability the probability of there being m microorganism

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at time t+ δ t is given by Feller.W[8]

P(m,t+ δ t)=(m-1)α P(m-1,t) δ t + (m+1) β P(m+1,t) δ t +(1-mα δ t- mβ δ t)P(m,t) (1)

As t $\rightarrow$ 0, the equation (1) becomes

$$\frac{\partial P(m,t)}{\partial t} = (m-1)\,\alpha\, P(m-1,t) + (m+1)\,\beta\, P(m+1,t) - m(\alpha+\beta)P(m,t) \quad (2)$$

Suppose

$$\phi(z,t) = \sum_{k=0}^{\infty} P(k,t) z^k \quad (3)$$

be the generating function of P(k, t).

Differentiating equation (3) partially with respect to time variable t, we have

$$\frac{\partial \phi(z,t)}{\partial t} = \sum_{k=0}^{\infty} P'(k,t) z^k \quad (4)$$

where

$$P'(k,t) = \frac{\partial P(k,t)}{\partial t} \quad (5)$$

Replacing the symbol m by n in equation (2), multiplying it by z^n and taking the sum from o to ∞ with P(0, t)=0 and P(-1, t)=0, we get

$$\sum_{n=0}^{\infty} P'(n,t) z^n = \sum_{n=0}^{\infty} (n-1)\alpha P(n-1,t) z^n + \sum_{n=0}^{\infty} (n+1)\beta P(n+1,t) z^n - \sum_{n=0}^{\infty} n(\alpha+\beta) P(n,t) z^n \quad (6)$$

It can be shown that

$$\sum_{k=0}^{\infty} (k-1)\alpha P(k-1,t) z^k = \alpha z^2 \frac{\partial \phi}{\partial z}$$

$$(\alpha+\beta) \sum_{k=0}^{\infty} k P(k,t) z^k = (\alpha+\beta) z \frac{\partial \phi}{\partial z}$$

$$\sum_{k=0}^{\infty} (k+1)\beta P(k+1,t) z^k = \beta \frac{\partial \phi}{\partial z}$$

In view of the above equations the equation (6) becomes

$$(\alpha z - \beta)(z-1)\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial t} = 0 \quad (7)$$

The obtained partial differential equation is a Lagrange's equation. The general solution of the equation (7) is given by

$$\phi(z,t) = f\left(\frac{\beta - \alpha z}{1-z} e^{-(\alpha-\beta)t}\right) \quad (8)$$

where, f is an arbitrary function. The arbitrary function can be obtained using the initial condition. Suppose the number of bacteria present at time t=0 is r, we have

$$\phi(z,0) = z^r \quad (9)$$

From the equations (8) and (9), we have

$$z^r = f\left(\frac{\beta - \alpha z}{1-z}\right) \quad (10)$$

Introducing $\frac{\beta - \alpha z}{1-z} = u$, we have

$$f(u) = \left(\frac{\beta - u}{\alpha - u}\right)^r \quad (11)$$

Thus, the generating function $\phi(z,t)$ is given by

$$\phi(z,t) = \left[\frac{\beta - \left(\frac{\beta - \alpha z}{1-z} e^{-(\alpha-\beta)t}\right)}{\alpha - \left(\frac{\beta - \alpha z}{1-z} e^{-(\alpha-\beta)t}\right)}\right]^r \quad (12)$$

Note that $\phi(z,0) = z^r$, hence P(r,0)=1. Now, P(n, t) (n $\neq 0$) is the coefficient of z^n in the right hand side of equation (12) and it is given by

$$P(n,t) = \frac{b_1^r b_2^n}{b_2^r a_2^n}\left[\begin{array}{l} C(r,1)sC(n,n) + C(r,2)s^2 C(n+1,n) + C(r,3)s^3 C(n+2,n) \\ + \ldots\ldots + C(r,r)s^r C(n+r-1,n) \end{array}\right] \quad (13)$$

$$P(0,t) = \frac{a_1^m}{a_2^m} \quad (14)$$

Where, $b_1 = \beta - \alpha e^{-(\alpha-\beta)t}$,

$b_2 = \alpha(1 - e^{-(\alpha-\beta)t})$,

$a_1 = \beta(1 - e^{-(\alpha-\beta)t})$, $a_2 = \alpha - \beta e^{-(\alpha-\beta)t}$ and

$s = \frac{a_1 b_2 - a_2 b_1}{a_2 b_1}$

Using the equations (13) and (14) one can find the probability of there being n (n $\neq 0$) and n=0 bacteria at given time t respectively. It can be used to determine the time where the probability is maximum to have n bacteria. If the probability is one or near to one then there is possibility of having n bacteria at this time.

Numerical Work:

Case 1: Assuming α=0.1 and β=0.2 the various probabilities of there being n=30 bacteria at different times are computed when initially 3 bacteria present at t=0. These are shown in Fig.1.

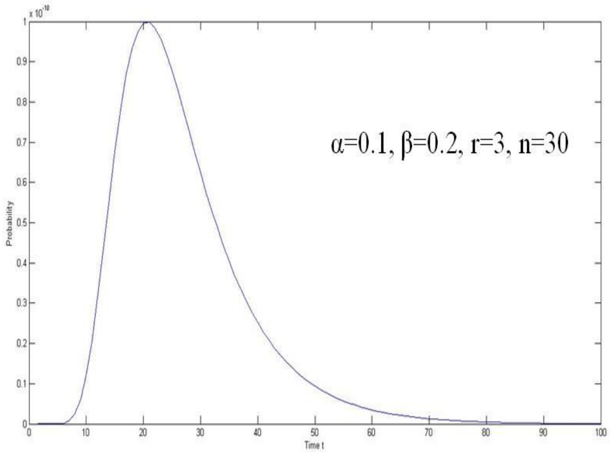

Fig.1. Probabilities with α=0.1, β=0.2, r=3, n=30

Case 2: For α=0.1 and β=0.3 the various probabilities of there being n=30 bacteria at different times are computed when initially 3 bacteria present at t=0. These are shown in Fig.2.

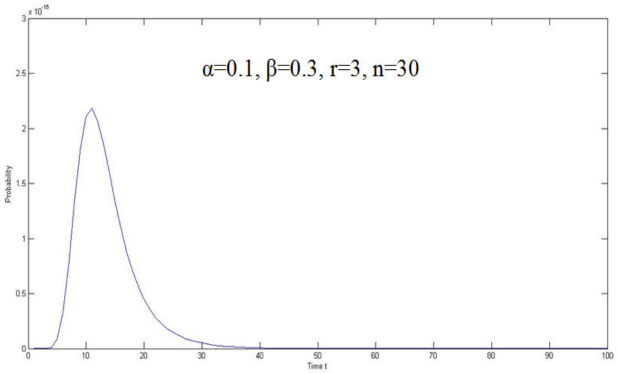

Fig.2. Probabilities with α=0.1, β=0.3, r=3, n=30

Case 3: Assuming α=0.1 and β=0.5 the various probabilities of there being n=30 bacteria at different times are computed when initially 3 bacteria present at t=0. These are shown in Fig.3.

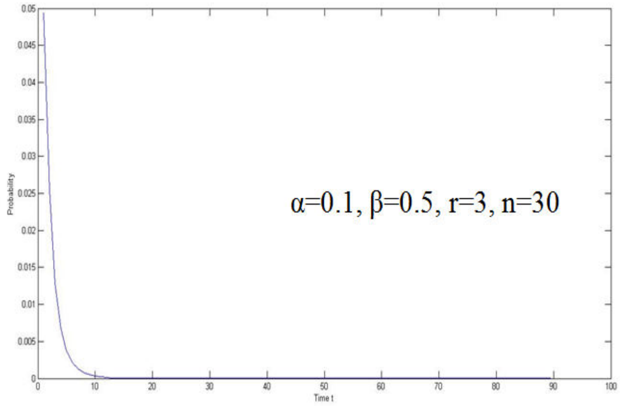

Fig.3. Probabilities with α=0.1,β=0.5, r=3, n=30

Case 4: Assuming α=0.6, β=0.4 and the probabilities of there being n=30 bacteria at different times are computed when initially r=3,4,5,6 bacteria present at t=0. These are shown in Fig4.

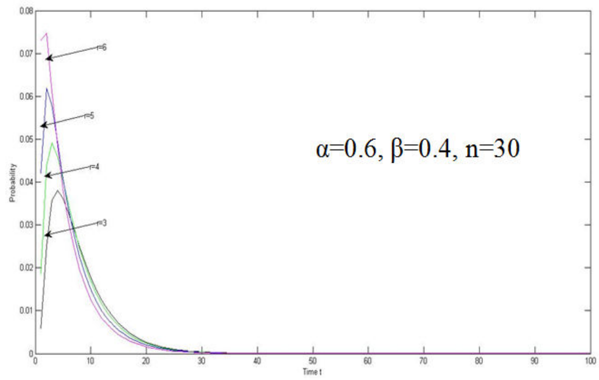

Fig.4. Probabilities with α=0.6, β=0.4, n=30

Case 5: For fixed values of α=2/10, β=1/10, r=3 and n=4, 5, 6, 7 the probabilities are computed and they are shown in Fig.5.

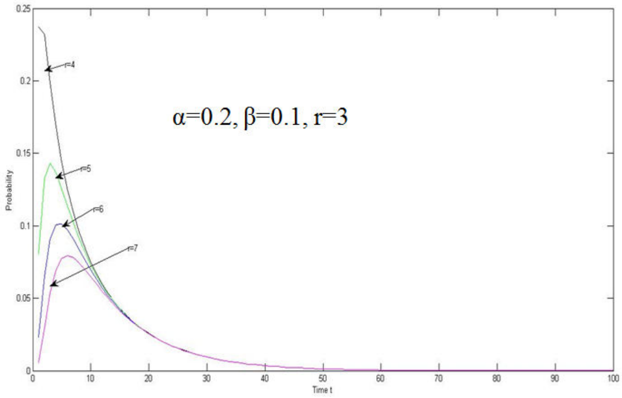

Fig.5. Probabilities with α=0.2, β=0.1, r=3

Case 6: For fixed values of α=1/10, β=2/10, r=3 and n=4, 5, 6, 7 the probabilities are computed and they are shown in Fig.6.

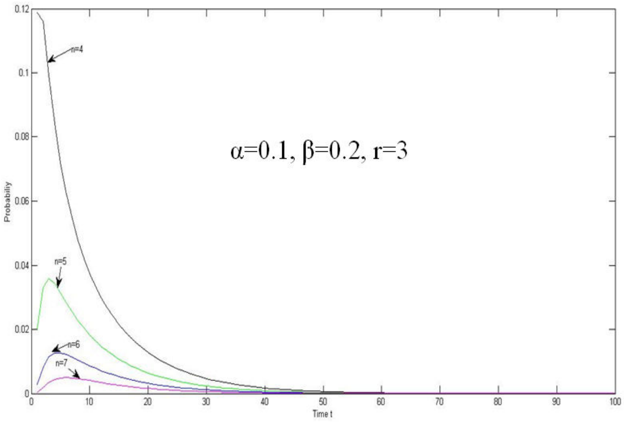

Fig.6. Probabilities with α=0.1, β=0.2, r=3

Case 7: For fixed values of α=0.1, r=3 and n=5 and β=0.2,0.25,0.3 ,0.35 the probabilities are computed and they are shown in Fig.7.

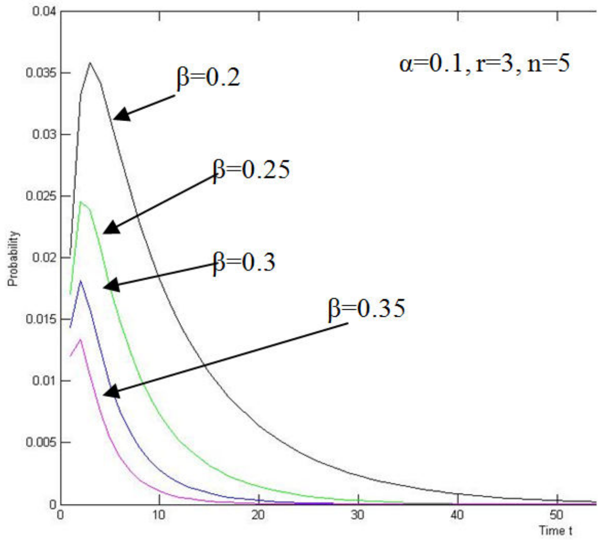

Fig.7. Probabilities with α=0.1, r=3, n=5

Case 8: For fixed values of β=0.1, r=3 and n=5 and α=0.4, 0.45, 0.5 ,0.55 the probabilities are computed and they are shown in Fig.8.

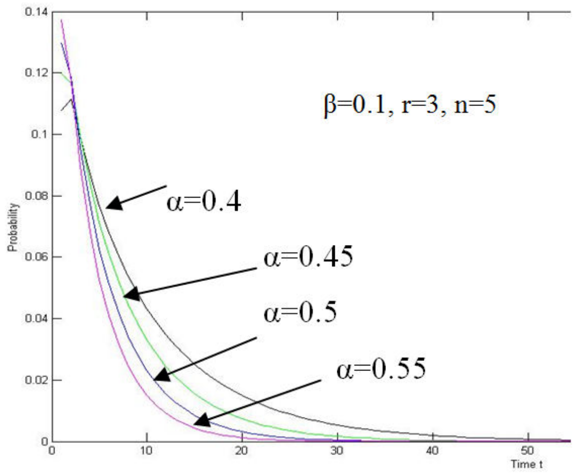

Fig.8. Probabilities with β=0.1, r=3, n=5

Case 9: For fixed values of α=0.8, β=0.4, r=10 the probabilities of there being bacteria n=20, 30, 40, 50 are computed and they are shown in Fig.9.

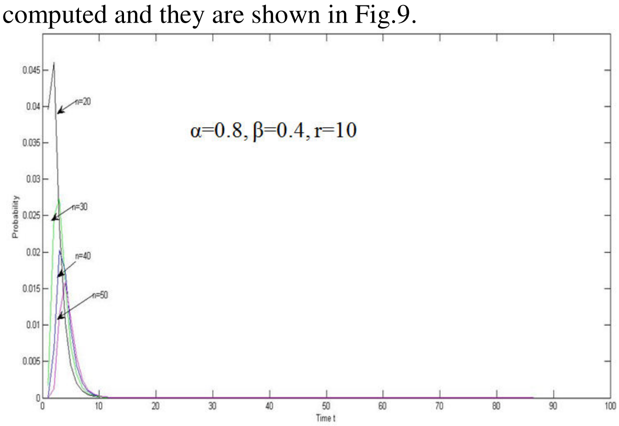

Fig.9. Probabilities with α=0.8, β=0.4, r=10

Case 10: For fixed values of α=0.1, β=0.3 and r=10 and the probabilities of there being bacteria n=0 are computed for various time values and they are shown in Fig.10.

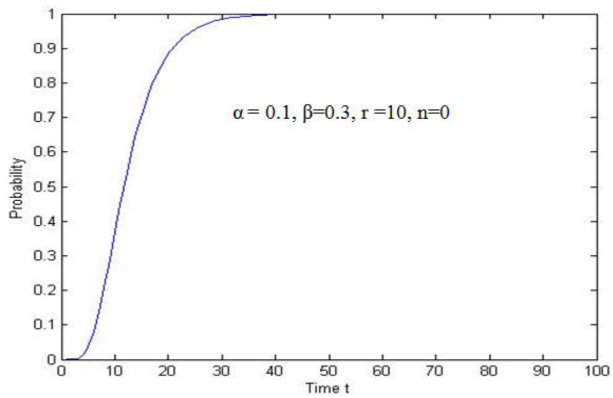

Fig.10. Probabilities with α=0.1, β=0.3, r=10, n=0

III. Conclusion

To know the influence of α and β on the probabilities of population of size n number of bacteria we have drawn the graphs using the MATLAB software and a program by assuming certain values to n, r, α and β. If α=0.1, β=0.2, r=3 and n=30 it is observed that the probabilities are increasing in the time interval (0, 20) and decreasing for t greater than 20 (Fig.1). When the β is increased from 0.2 to 0.3 the time interval of increasing probabilities is reduced (Fig.2). When β>>α. the probabilities are decreasing with increase in time to have 30 bacteria when initially there is a 3 bacteria (Fig.3). For fixed values of α, β, n the various probabilities are computed when there is initially 3,4,5,6 numbers of bacteria and it is observed the probabilities are increasing (Fig.4). Similarly for fixed values of α, β and r the probabilities to have 4,5,6,7 number of bacteria a are decreasing in the time interval (0, 7) as n increasing (Fig.5). By reversing the values of α and β of the case 5 , it is observed that the probabilities are reduced (Fig.6). For fixed values of α, r ,n and increasing values of β the probabilities are decreasing in each case(Fig.7). A similar situation is noticed when β, r, n valued are fixed and the values of α are increasing(Fig.8). When n is increasing for fixed values of other, the probabilities are decreasing (Fig.9).For α=0.1, β=0.3 and r=10 the probability to have zero bacteria is one at time t=38 (Fig.10).

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