Rotation Surfaces in 4-Dimensional Pseudo-Euclidean Spaces

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Abstract – In this paper, we study the rotation surfaces in 4-dimensional pseudo-Euclidean spaces \( E_1^4 \) and \( E_2^4 \). We investigate the rotation surfaces and also give a classification of rotation surfaces in \( E_1^4 \) under the condition \( (\Psi^I)_{uv} + (\Psi^I)_{vu} = \lambda \Psi^I, \lambda \in \mathbb{R}, I = 1, 2, 3, 4 \). Furthermore, we examine the isothermal rotation surfaces in \( E_2^4 \) and give some characterizations for them.

Keywords – Rotation Surface, Mean Curvature, Isotothermal Surface, Weingarten Surface, Christoffel Symbols.

I. INTRODUCTION

The geometry of rotation surfaces has been studied widely in Euclidean space \( E^3 \) as well as Lorentz-Minkowski space \( E_1^3 \). It is well known that, induced metric on a surface \( M \) in \( E_1^3 \) can be non-degenerate or degenerate. If the induced metric is non-degenerate, then \( M \) is called a semi-Riemannian surface and otherwise \( M \) is called a degenerate surface.

Relative to Takahashi’s theorems [16] for minimal submanifolds, the idea of submanifolds of finite type in Euclidean space has been introduced by B.Y. Chen [2]. As a generalization of Takahashi’s theorem for the case of hypersurfaces, O.J. Garay [9] has considered the hypersurfaces satisfying the condition \( \Delta x = Ax \), where \( x \) is an isometric immersion from \( M \) to \( \mathbb{R}^{n+1} \), \( \Delta \) is the Laplacian on \( M \) and \( A \) is an \((n+1)\)-dimensional diagonal matrix and the theory is recently greatly developed. D.W. Yoon has studied the translation surfaces in the 3-dimensional Minkowski space whose Gauss map \( G \) satisfies the condition \( \Delta G = AG, A \in \text{Mat}(3, \mathbb{R}) \), where \( \text{Mat}(3, \mathbb{R}) \) denotes the set of \( 3 \times 3 \) real matrices [17].

O.J. Garay [10] has investigated the rotation surfaces, in Euclidean space \( E^3 \), whose component functions are eigenfunctions of its Laplacian and he saw that these surfaces must be a Catenoid, a sphere or a right circular cylinder.

Also, the Lorentz version of the non-degenerate surfaces \( M_s^2 \), with index \( s = 0,1 \) in \( E_1^{m+1} \), have been classified under the condition \( \Delta H = \lambda H \) by A. Ferrandez and P. Lucas in [8]. They have proved that, \( M_s^2 \) is a zero mean curvature surface everywhere, either an open piece of a B-scroll surface or an open piece of the surfaces \( S^2(r) \times \mathbb{R}, H^2(r) \times \mathbb{R}, S_1^2(r) \times \mathbb{R}, H^2(r) \times S_1^2(r) \).

In [12], G. Kaimakamis and B. Papantoniou have classified the rotation surfaces without parabolic points, in the 3-dimensional Lorentz-Minkowski space, under the condition

\[
\Delta'^I \tilde{r} = A'I, 
\]

where \( \Delta'^I \) is the Laplace operator with respect to the second fundamental form and \( A \) is a real \( 3 \times 3 \) matrix. They have proved that, such surfaces are either minimal or Lorentz hyperbolic cylinders or pseudospheres of real or imaginary radius.

Recently, the rotation surfaces with no zero Gaussian curvature \( K_G \) in the 3-dimensional Lorentz-Minkowski space have been classified under the condition

\[
\Delta x^I = \lambda x^I, 
\]

where \( \Delta \) is the Laplace operator with respect to the induced metric and \( \lambda \) is real number, by M. Bekkar and H. Zoubir in [1]. They have proved that such surfaces are either minimal or Lorentz hyperbolic cylinders or circular cylinders or pseudo-spheres of real radius or pseudo-hyperbolic space of imaginary radius.

On top of all this, the rotation surfaces have been studied in four-dimensional Euclidean space and pseudo-Euclidean space. In 1993, L. HuiLi and L. GuiLi have constructed a class of rotation surfaces with constant mean curvature in 4-dimensional pseudo-Euclidean space \( E_2^3 \) [11]. Y.H. Kim and D.W. Yoon [13] have studied the rotation surfaces in \( E_2^3 \), too. They have obtained the complete classification theorems for flat rotation surfaces with finite type Gauss map, pointwise 1-type Gauss map and an equation in terms of the mean curvature vector.

In [7], U. Dursun has classified the rotation hypersurfaces of Lorentz-Minkowski space with pointwise 1-type Gauss map of the first and the second kind. The invariant theory of surfaces in the four-dimensional Euclidean space to the class of general rotation surfaces with meridians lying in two-dimensional planes has been introduced in [14].

In this article, firstly we study the rotation surfaces in 4-dimensional pseudo-Euclidean space \( E_1^4 \) which is given by the vector-valued function

\[
\Psi_1(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u) \cos v, g(u) \sin v). 
\]

After, we give a classification of rotation surfaces in \( E_2^4 \) under the condition \( (\Psi^I_1)_{uv} + (\Psi^I_1)_{vu} = \lambda_1 \Psi^I_1 \). And finally, we consider the isothermal rotation surfaces in 4-dimensional pseudo-Euclidean space \( E_2^4 \) which is given by

\[
\Psi_2(u, v) = (f(u) \cosh v, f(u) \sinh v, g(u) \cos v, g(u) \sin v), 
\]

and give some characterizations about these surfaces.

II. PRELIMINARIES

Let \( E_1^m \) denote the \( m \)-dimensional pseudo-Euclidean space with signature \((s, m-s)\), that is, the real vector space \( \mathbb{R}^m \) endowed with the metric \( \langle \cdot, \cdot \rangle \) which is defined by

\[
\langle \cdot, \cdot \rangle = -\sum_{i=1}^{s} (dx_i)^2 + \sum_{i=s+1}^{m} (dx_i)^2, 
\]

where \((x_1, x_2, \ldots, x_m)\) is a standart rectangular coordinate system in \( E_1^m \). A vector \( \nu \) of \( E_1^m \) is said to be

i) spacelike, if \((\nu, \nu) > 0 \),

ii) timelike, if \((\nu, \nu) < 0 \) and

iii) null (or lightlike), if \((\nu, \nu) = 0 \).
The pseudo-Riemannian sphere $\mathbb{S}^m_{r}(x_0, r)$ centered at $x_0 \in E^m_1$ with radius $r > 0$, of $E^m_1$ is defined by
$\mathbb{S}^m_{r}(x_0, r) = \{ x \in E^m_1 : (x - x_0, x - x_0) = r^2 \}$
and the pseudo-hyperbolic space $\mathbb{H}^{m-1}(x_0, r)$ centered at $x_0 \in E^m_1$ with radius $r > 0$, of $E^m_1$ is defined by
$\mathbb{H}^{m-1}(x_0, r) = \{ x \in E^m_1 : (x - x_0, x - x_0) = -r^2 \}$.

The pseudo-Riemannian sphere $\mathbb{S}^m_{r}(x_0, r)$ is diffeomorphic to $\mathbb{R}^m \times \mathbb{S}^{m-1}$ and the pseudo-hyperbolic space $\mathbb{H}^{m-1}(x_0, r)$ is diffeomorphic to $\mathbb{R}^m \times \mathbb{H}^{m-2}$. The hyperbolic space $\mathbb{H}^{m-1}(x_0, r)$ is defined by
$\mathbb{H}^{m-1}(x_0, r) = \{ x \in E^m_1 : (x - x_0, x - x_0) = -r^2, x_1 > 0 \}$.

Let $\Psi : M^m \rightarrow E^m_1$ be an isometric immersion of an oriented $m$-dimensional pseudo-Riemannian submanifold $M^m$ into $E^m_1$. Henceforward, a submanifold in $E^m_1$ always means pseudo-Riemannian. Let $\nabla$ be the Levi-Civita connection of $E^m_1$ and $\nabla$ be the induced connection on $M^m$. Then, for any vector fields $X, Y$ tangent to $M^m$, we have the Gauss formula
$$\nabla_X Y = \nabla_X Y + h(X, Y),$$
where $h$ is the second fundamental form which is symmetric in $X$ and $Y$. For a unit normal vector field $\xi$, the Weingarten formula is given by
$$\nabla_X \xi = -A_X \xi,$$
where $A_X$ is the Weingarten map or the shape operator with respect to $\xi$. The Weingarten map $A_X$ is a self-adjoint endomorphism of $TM$ which cannot be diagonalized generally. It is known that, $h$ and $A_X$ are related by
$$h(X, Y, \xi) = (A_X Y, \xi).$$

The covariant derivative $\nabla h$ of the second fundamental form is defined by
$$\nabla h(X, Y, Z) = \nabla_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$
where $\nabla$ denotes the linear connection induced on the normal bundle $TM$. Then, the Codazzi equation is given by
$$\nabla_X h(Y, Z) = \nabla_Y h(X, Z).$$
Let $e_1, e_2, \ldots, e_m$ be a local orthonormal frame field in $E^m_1$ such that $e_1, \ldots, e_n$ are tangent to $M^m$ and $e_{n+1}, \ldots, e_m$ are normal to $M^m$. Let $w_1, w_2, \ldots, w_m$ be the coframe of $e_1, e_2, \ldots, e_m$. We shall make use of the following convention on the ranges of indices $1 \leq i, j, \ldots, n, n + 1 \leq s, \ldots, m$, $1 \leq A, B, \ldots, m$. Then, $w_A(e_B) = \delta_{AB}$ and the pseudo-Riemannian metric on $E^m_1$ is given by
$$ds^2 = \sum_{i=1}^{n} e_A w_A^2 + e_A \left( e_A, e_A \right) = \pm 1.$$

Let $w_A$ be the dual 1-form defined by $w_A(X) = (e_A, X)$. Furthermore, the connection forms $w_{AB}$ are defined by
$$de_A = \sum_{i=1}^{n} w_{AB} e_B w_A, \quad w_{AB} + w_{BA} = 0.$$
Then, the structure equations of $E^m_1$ are obtained as follows:
$$dw_A = \sum_{i=1}^{n} e_B w_{AB} \wedge w_B, \quad w_{AB} + w_{BA} = 0.$$ (1)

The canonical forms $\{w_A\}$ and the connection forms $\{w_{AB}\}$ restricted to $M^m$ are also denoted by the same symbols. Then, we have
$$w_A = 0, \quad s = n + 1, \ldots, m.$$ (4)

Since $w_A$ are zero forms on $M^m$, there are symmetric tensor $h^A_{B}$ by Cartan’s lemma such that
$$w_{iA} = \sum_{j=1}^{n} e_j h^A_{B} w_j, \quad h^A_{B} = h^A_{B}.$$ (5)

The mean curvature vector $H$ of $M^m$ in $E^m_1$ is given by
$$H = \frac{1}{n} \sum_{r=s+1}^{m} \sum_{A=1}^{n} e_A h^A_{B} e_A.$$ (6)

Also, the covariant differentiation of $e_i$ is written by
$$d e_i = \sum_{A=1}^{n} e_A w_{iA} e_A \wedge w_A + w_{iA} e_A \wedge w_A.$$ (7)

Let denote by $E, F, G$ the coefficients of the first fundamental form of $M^m$. If $\Psi(u, v)$ is a smooth function, the second differential parameter of Laplacian (Beltrami) of a function $\Psi(u, v)$ with respect to the first fundamental form of $M^m$ is the operator $\Delta$ which is defined by
$$\Delta \Psi = -\frac{1}{\sqrt{(g_{uv} - g_{uu})}} \left( \frac{g_{uv} - g_{uu}}{\sqrt{|g_{uv} - g_{uu}|}} \right)_{\Psi}.$$ (8)

### III. Rotation Surfaces in $E^4_1$

Firstly, let us recall the notion of rotation surfaces in $E^4_1$. Let $C$ be a curve in $span\{e_1, e_2, e_3\}$ parametrized by arclength
$$z(u) = (f(u), g(u), \rho(u), 0), u \in I,$$ where $\rho(u) > 0$. The orbit of $C$ under the action of the orthogonal transformations of $E^4_1$ leaving the plane $Oxy$,
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \nu & -\sin \nu \\ 0 & 0 & \sin \nu & \cos \nu \end{bmatrix}, \nu \in \mathbb{R},$$
is a surface given by $M_1(u, v) = (f(u), g(u), \rho(u) \cos \nu, \rho(u) \sin \nu), u \in I, \nu \in [0, 2\pi]$. Then, $M_1(u, v)$ is called a rotation surface in $E^4_1$. That means, $M_1(u, v)$ is orbit of a curve by rotating it around a plane.

We also have the another kind of rotation surface in $E^4_1$, it is the orbit of a plane curve rotated around both two planes. This surface is defined as following:

Let $C$ be a regular curve in $span\{e_1, e_3\}$ parametrized by arc-length
$$r(u) = (f(u), 0, g(u), 0), u \in I$$
and
$$B = \begin{bmatrix} \cos \alpha v & -\sin \alpha v & 0 & 0 \\ \sin \alpha v & \cos \alpha v & 0 & 0 \\ 0 & 0 & \cos \beta v & -\sin \beta v \\ 0 & 0 & \sin \beta v & \cos \beta v \end{bmatrix}, \nu \in \mathbb{R}_*,$$ is a subgroup of the orthogonal transformations group on $E^4_1$, where $\alpha, \beta$ are positive constants and $f^2(u) + g^2(u) \neq 0$.

The orbit of $C$ under the action of the subgroup $B$ is a surface in $E^4_1$ given by $M_2(u, v) = (f(u) \cos \nu, f(u) \sin \nu, g(u) \cos \beta v, g(u) \sin \beta v)$, which is called general rotational surface whose meridian lies in two dimensional planes. Then $r(u)$ is called meridian and $\alpha, \beta$ are called rates of rotation [4].

Now, we’ll study the general rotation surfaces in 4-dimensional pseudo-Euclidean space $E^4_1$. Here, $E^4_1$ is given with the pseudo-Riemannian metric
$$\langle \cdot, \cdot \rangle = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2,$$
where \((x_1, x_2, x_3, x_4)\) is a standard rectangular coordinate system in \(E^4\). Let \(x(u) = (x_1(u), x_2(u), x_3(u), x_4(u)); u \in J \subset \mathbb{R}\) be a smooth curve in \(E^4\) and

\[
\mathcal{A} = \begin{bmatrix}
\cosh v & \sinh v & 0 & 0 \\
0 & 0 & \cosh v & \sinh v \\
0 & 0 & -\sinh v & \cosh v \\
0 & -\sinh v & \cosh v & 0
\end{bmatrix}, \quad v \in \mathbb{R},
\]

be a subgroup of the orthogonal transformations group on \(E^4\), where \(J\) is an open interval and \(\alpha, \beta\) are constants. Then, a general rotation of the meridian curve \(x(u)\) in \(E^4\) is given by

\[
\Psi(u, v) = (\Psi^1(u, v), \Psi^2(u, v), \Psi^3(u, v), \Psi^4(u, v)),
\]

where

\[
\Psi^1(u, v) = x_1 \cosh v + x_2 \sinh v,
\]

\[
\Psi^2(u, v) = x_1 \sinh v + x_2 \cosh v,
\]

\[
\Psi^3(u, v) = x_3 \cos v - x_4 \sin v,
\]

\[
\Psi^4(u, v) = x_3 \sin v + x_4 \cos v.
\]

Now, we consider the rotation surface \(\Psi_1\) in \(E^4\), which is defined by the vector-valued function

\[
\Psi_1(u, v) = (f(\cosh u, \sinh u, g(u) \cos v, g(u) \sin v), \quad u \in J \subset \mathbb{R}, \quad v \in [0, 2\pi]),
\]

where \(f(\cosh u, \sinh u, g(u) \cos v, g(u) \sin v)\) and \(g(u)\) are nonzero smooth functions and the meridian curve \(x(u) = (f(u), 0, g(u), 0)\) is given by the arc-length, that is

\[
f^{-2}(u) + g^{-2}(u) = 1, \quad \forall u \in J \subset \mathbb{R}.
\]

For this rotation surface, the coefficients of the first fundamental form are

\[
E = f^{-2} + g^{-2} = 1, \quad F = 0, \quad G = f^{-2} + g^{-2}.
\]

We choose the moving frame \(e_1, e_2, e_3, e_4\)

\[
e_1 = \frac{1}{\sqrt{f^2 + g^2}}(f \sinh v, f \cosh v, g \cos v, g \sin v),
\]

\[
e_2 = \frac{1}{\sqrt{f^2 + g^2}}(g \cosh v, g \sinh v, f \cos v, f \sin v),
\]

\[
e_3 = \frac{1}{\sqrt{f^2 + g^2}}(g \sinh v, g \cosh v, f \sin v, -f \cos v),
\]

where \(e_1, e_2\) are tangent to \(\Psi_1\) and \(e_3, e_4\) are normal to \(\Psi_1\). Then, one can easily see that

\[
\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = \langle e_3, e_3 \rangle = 1, \quad \langle e_3, e_4 \rangle = -1.
\]

Furthermore, we have

\[
w_1 = du, \quad w_2 = \sqrt{f'^2 + g'^2}dv
\]

and so \(e_1, e_2\) can be written as

\[
e_1 = \frac{d}{du}, \quad e_2 = \frac{1}{\sqrt{f^2 + g^2}} \frac{d}{dv}.
\]

Using (3), (4), (12), (13) and (14), we obtain the following coefficients of the second fundamental form \(h\) and the connection forms \(\mathcal{W}_{AB}\):

\[
h_{11} = g^{-2} f' g'' + f g^{-2} \frac{g''}{f'},
\]

\[
h_{12} = \frac{1}{f^2 + g^2}(f' g - f g'),
\]

\[
h_{22} = -\frac{1}{f^2 + g^2}(f g' + f' g),
\]

\[
h_{13} = h_{12} = 0,
\]

\[
h_{23} = h_{22} = 0,
\]

\[
h_{14} = h_{24} = \frac{1}{f^2 + g^2}(f' g - f g')
\]

and

\[
w_{12} = \frac{1}{f^2 + g^2}(f f' + g g')w_2, \quad w_{13} = (f g'' - f' g)w_1,
\]

\[
w_{14} = \frac{1}{f^2 + g^2}(f g' - f' g)w_2, \quad w_{23} = -\frac{1}{f^2 + g^2}(f' g + f g')w_2,
\]

\[
w_{24} = \frac{1}{f^2 + g^2}(f' g - f g')w_1, \quad w_{34} = \frac{1}{f^2 + g^2}(g g' - f f')w_2.
\]

Also, from (7), (13), (14), (15), (16) and (17) we get

\[
\Psi_{e_1} \Psi_{e_2} \Psi_{e_3} = \left(\begin{array}{c}
(f' f'' + g g'')(e_2 + f' g g'')e_3 \\
\frac{1}{f^2 + g^2}((f' f')g + (g' g')f)e_3 \\
\frac{1}{f^2 + g^2}((f' f')g + (g' g')f)e_3 \\
(e_2 - f' f'')f g + (g' g')f''e_3
\end{array} \right)
\]

From (6) and (11), the mean curvature vector \(H\) of the rotation surface \(\Psi_1\) given by (9) is

\[
H = \frac{1}{f^2 + g^2} \sum_{i=1}^{4} \varepsilon_i e_i h_{ii} e_3 = \frac{1}{f^2 + g^2}(f' f')g + (g' g')f + \frac{1}{f^2 + g^2}(g g' - f f')e_3.
\]

where \(A = f f' f'' + g g' + f f'' + f' g g' + f' g'\) and

\[B = g g' f' + g f' g' + g g'' + f f' g' - g g'.\]

Thus, we have the following corollary which is a particular case of the well-known Beltrami’s formula \(\Delta x = -nH\), which is valid for any isometric immersion in any pseudo-Euclidean space \(M^n \rightarrow \mathbb{R}^n\) [3]:

**Corollary 1.** Let \(\Psi_1\) be a rotation surface which is given by (9) in \(E^4\). Then the mean curvature vector \(H\) of \(\Psi_1\) is

\[
H = -\frac{1}{2} \Delta \Psi_1,
\]

where \(\Delta \Psi_1\) denotes the Laplacian of the rotation surface \(\Psi_1\).

**Proof.** From (8), the Laplacian \(\Delta\) of the rotation surface \(\Psi_1\) given by (9) is

\[
\Delta \Psi_1 = -\frac{1}{f^2 + g^2}(A \sinh v, A \sinh v, B \cosh v, B \sinh v).
\]

If we compare (19) and (20), then we get \(2H = -\Delta \Psi_1\) and this ends the proof.

Furthermore, in terms of components of the meridian curve, the mean curvature of the rotation surface \(\Psi_1\) is obtained as

\[
H = \frac{1}{f^2 + g^2}(f' f' g + f g' f' + g g' g')
\]

Since the meridian curve of the rotation surface (9) is given by arc-length, from (10) there exists a nonzero smooth function \(t = t(u)\) such that

\[
f(u) = \sinh t(u), \quad g(u) = \cosh t(u), \quad \forall u \in J \subset \mathbb{R}.
\]

Thus, we can give the following:

**Proposition 1.** If the rotation surface \(\Psi_1\) given by (9) in \(E^4\) is minimal, then

\[
t' = -\frac{(f g)'}{f^2 + g^2}.
\]

**Proof.** It is trivial from (21) and (22).
The Gaussian curvature \( K \) of the rotation surface \( \Psi_1 \) given by (9) is obtained as
\[
K = \sum_{i=3}^{4} e_i (h^2_i h^2_i - h^2_i h^2_i) \tag{23}
\]
\[
= \frac{1}{(f'^2 + g'^2)^2} ((f'' g' - f' g')(f g' + f g')(f^2 + g^2) - (f' g' - f g')^2).
\]
Thus, from (23) we can give the following corollary:

**Corollary 2.** Let \( \Psi_1 \) be a rotation surface which is given by (9) in \( E_4^1 \). If \( \frac{f(u)}{g(u)} = c, c \in \mathbb{R} \), then \( \Psi_1 \) is a flat surface.

**IV. A CLASSIFICATION OF ROTATION SURFACES IN \( E_4^1 \)**

In this section, we classify the rotation surface \( \Psi_1 \) given by (9) in \( E_4^1 \) under the condition
\[
(\Psi_1(u,v))_{uu} + (\Psi_1(u,v))_{vv} = \lambda_1\Psi_1(u,v), \tag{24}
\]
where \( \lambda_1 \in \mathbb{R}, i = 1,2,3,4 \).

Here, we again suppose that the meridian curve \( x(u) = (f(u),0,0,0) \) is given by the arc-length, that is
\[
f' = \sinh t, \quad g' = \cosh t, \quad \forall u \in \mathbb{R}, \tag{25}
\]
where \( t(u) \neq 0 \).

If we apply the condition (25) to the rotation surface (9), then we have the system of equations
\[
\begin{align*}
\dot{f}' + f &= \lambda_1 f, \\
\dot{f}' + f &= \lambda_2 f, \\
\dot{g}' - g &= \lambda_3 g, \\
\dot{g}' - g &= \lambda_4 g.
\end{align*}
\]
\[
(26)
\]
Thus, we have \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4. \)
If we put \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \mu \), we get the system of equations
\[
\begin{align*}
\dot{f}' + f &= \lambda f, \\
\dot{g}' - g &= \mu g.
\end{align*}
\]
\[
(27)
\]
By using (25), we obtain the following system of equations:
\[
\begin{align*}
\cosh t' &= (\lambda - 1) f, \\
\sinh t' &= (\mu + 1) g.
\end{align*}
\]
\[
(28)
\]
So, the problem of classifying the rotation surfaces \( \Psi_1 \) given by (9) which satisfies the condition (24) is reduced to the integration of system of ordinary differential equations (28).

Now, let us examine the system of equations (28) according to the values of the constants \( \lambda \) and \( \mu \):

**A. \( \lambda = 1, \mu = -1 \)**:

Then, the system of equations (28) reduces to
\[
\begin{align*}
\cosh t &= 0, \\
\sinh t &= 0.
\end{align*}
\]
But this is not possible. So, in this case, there are no rotation surfaces which are given by (9) in \( E_4^1 \).

**B. \( \lambda = 1, \mu \neq -1 \)**:

In this case, we have \( \cosh t = 0 \). This is not possible because of the definition of the function \( \cosh \). Thus, there are no rotation surfaces for this case, too.

C. \( \lambda \neq 1, \mu = -1 \):

Since \( \sinh t = 0 \), we get \( f(u) = \sinh t = 0 \) and so \( f(u) = c_1, c_2 \in \mathbb{R} \). Therefore, from (10), we obtain
\[
g(u) = \pm u + c_2, c_2 \in \mathbb{R} \text{. Thus, when } \lambda \neq 1 \text{ and } \mu = -1 \text{ the rotation surface can be parametrized by}
\]
\[
\Psi_1(u,v) = (c_1 \cosh v, c_1 \sinh v, (\pm u + c_2) \cos v, (\pm u + c_2) \sin v),
\]
\( c_1, c_2 \in \mathbb{R} \).

**D. \( \lambda \neq 1, \mu \neq -1 \)**:

Now, let us solve the system of ordinary differential equations (28).

In this system of equations, if the first equation is multiplied by \( \sinh t \) and the second equation is multiplied by \( -\cosh t \) and add up the resulting equations, we get the equation
\[
(\lambda - 1)f \sinh t - (\mu + 1)g \cosh t = 0.
\]
By using (25), we have
\[
(\lambda - 1)ff' - (\mu + 1)gg' = 0
\]
or equivalently
\[
ff' - \frac{\lambda + 1}{\lambda - 1}gg' = 0.
\]
If we integrate (29), we obtain
\[
f^2 - \mu + 1 = t^2 r^2, r \in \mathbb{R}.
\]
In the last equation if \( \lambda = \mu + 2 \), then the rotation surface is either pseudo-Riemannian sphere \( S^2_1(r) \) of real radius \( r \), given by the equation \( -x_1^2 + x_2^2 + x_3^2 + x_4^2 = r^2 \), or the hyperbolic space \( \mathbb{H}^3_1(r) \) with imaginary radius, given by the equation \( -x_1^2 + x_2^2 + x_3^2 + x_4^2 = -r^2 \).

Thus, we can give the following theorem:

**Theorem 1.** Let \( \Psi_1 \) be a rotation surface given by (9) in \( E_4^1 \). Then, the rotation surface \( \Psi_1 \) satisfies the condition (24) if and only if \( \Psi_1 \) is one of the following:

i) The rotation surface
\[
\Psi_1(u,v) = (c_1 \cosh v, c_1 \sinh v, (\pm u + c_2) \cos v, (\pm u + c_2) \sin v),
\]
\( c_1, c_2 \in \mathbb{R} \),

ii) the pseudo-Riemannian sphere \( S^2_1(r) \) of real radius \( r \) or the hyperbolic space \( \mathbb{H}^3_1(r) \) with imaginary radius.

**IV. ISOTHERMAL ROTATION SURFACES IN 4-DIMENSIONAL PSEUDO-EUCLIDEAN SPACES**

It is known that [6], a regular parametrized surface \( \Psi(u,v) \) is said to be isothermal if
\[
E = (\Psi_{u,v}) = (\Psi_{u,v}) = G, F = (\Psi_{u,v}) = 0.
\]
So, we get

**Theorem 2.** The rotation surface \( \Psi_1 \) given by (9) whose meridian curve is given by arc-length in \( E_4^1 \) can’t be isothermal.

**Proof.** For the rotation surface \( \Psi_1 \) given by (9), the coefficients of the first fundamental form are
\[
E = f^2 + g^2 = 1, \quad F = 0, \quad G = f^2 + g^2.
\]
If this surface is isothermal, then we get
\[
-f' + gg' = f^2 + g^2 = 1. \tag{31}
\]
But we can’t find \( f \) and \( g \) which satisfy (31). Thus, the rotation surface (9) whose meridian curve is given by arc-length in \( E_4^1 \) can’t be isothermal.
Hence, let us investigate the isothermal rotation surfaces in $E^3_2$. Let $E^3_2$ denote the 4-dimensional pseudo-Euclidean space with the pseudo-Riemannian metric $\langle \cdot, \cdot \rangle = -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 - (dx_4)^2$, where $(x_1,x_2,x_3,x_4)$ is a standard rectangular coordinate system in $E^3_2$.

Now, we consider the following rotation surface $\Psi_2$ in $E^3_2$:

$$\Psi_2(u,v) = (f(u)\cosh, f(u)\sinh, g(u)\cosh, g(u)\sinh),$$

(32) where $f$ and $g$ are nonzero smooth functions and the meridian curve $x(u) = (f(u), 0, g(u), 0)$ is given by the arc-length, such that

$$f^2(u) + g^2(u) = 1, \forall u \in I \subset \mathbb{R}.$$  
(33)

For the rotation surface (32)

$$E = -f^2 + g^2 = 1, F = 0, G = f^2 - g^2.$$  
(34)

If the rotation surface (32) is isothermal, then from (30) we get

$$E = -f^2 + g^2 = 1 = f^2 - g^2 = G.$$  
(35)

We choose the following moving frame $e_1, e_2, e_3, e_4$ such that $e_1, e_2$ are tangent to $\Psi_2$ and $e_3, e_4$ are normal to $\Psi_2$:

$$e_1 = (f'\cosh, f'\sinh, g'\cosh, g'\sinh)$$
$$e_2 = (f\sinh, f'\cosh, g\sinh, g'\cosh)$$
$$e_3 = (g\cosh, g'\sinh, f'\cosh, f\sinh)$$
$$e_4 = (g\sinh, g'\cosh, f\sinh, f'\cosh).$$
(36)

Then, we can easily get

$$\langle e_1, e_1 \rangle = 1, \langle e_3, e_3 \rangle = -1.$$  
(37)

Also, we have

$$w_1 = du, w_2 = dv$$
(38)

and so $e_3$ and $e_4$ can be written as

$$e_3 = \frac{\partial}{\partial u}, e_4 = \frac{\partial}{\partial v}.$$  
(39)

Using (3), (4), (36), (37) and (38) we get the following coefficients of the second fundamental form and the connection forms $\nabla$:

$$h_{11}^1 = f'g' - f''g, \quad h_{12}^1 = h_{21}^1 = 0, \quad h_{22}^1 = f'g' - f''g'$$
$$h_{11}^2 = h_{12}^2 = 0, \quad h_{22}^2 = f'g' - f''g.$$  
(40)

and

$$w_{12} = 0, w_{13} = (f'g' - f''g) w_1,$$
$$w_{14} = (f'g' - f''g) w_2,$$
$$w_{23} = (f'g' - f''g) w_1,$$
$$w_{24} = (f'g' - f''g) w_2,$$
(41)

Furthermore, from (7), (37), (38), (39), (40) and (41) we have

$$\nabla e_1 e_2 = (f'g' - f''g)e_3, \quad \nabla e_1 e_1 = (f'g' - f''g)e_4$$
$$\nabla e_1 e_2 = (f'g' - f''g)e_4, \quad \nabla e_1 e_3 = (f'g' - f''g)e_2$$
$$\nabla e_1 e_2 = (f'g' - f''g)e_3, \quad \nabla e_2 e_2 = (f'g' - f''g)e_2$$
$$\nabla e_1 e_3 = (f'g' - f''g)e_2, \quad \nabla e_2 e_3 = (f'g' - f''g)e_1.$$  
(42)

**Corollary 3.** The parameter curves of the isothermal rotation surface $\Psi_2$ given by (32) are geodesics on it.

From (6), the mean curvature vector $H$ of the isothermal rotation surface $\Psi_1$ given by (32) is

$$H = \frac{1}{2} \left[ \sum_{s=3}^4 \sum_{i=1}^2 e_i \varepsilon_h \varepsilon_s = \frac{1}{2} \left( f''g' - f'g'' + f'g' - f''g \right) \varepsilon_s \right.$$  

$$= \frac{1}{2} (C \cosh, C \sinh, D \cosh, D \sinh),$$

where $C = (f'g' - f''g)^2 + f'g' - f''g$ and $D = (f'g' - f''g)^2 + f'g' - f''g$.

Thus, from (35) we have

$$H = \frac{1}{2} (f'g' - f''g) \cosh, (f'g' - f''g) \sinh, (g' + g) \cosh,$$

(43)

Therefore, we can give the following theorem:

**Theorem 3.** Let the rotation surface $\Psi_2$ given by (32) be isothermal. Then the mean curvature vector $H$ of $\Psi_2(u,v)$ is

$$H = \frac{1}{2} \left( (\Psi_2(u,v))_{uu} + (\Psi_2(u,v))_{vv} \right).$$

We know that, the Laplacian $\Delta f$ of a differentiable function $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ is defined by $\Delta f = (\frac{\partial^2 f}{\partial u^2}) + (\frac{\partial^2 f}{\partial v^2})$, $(u,v) \in U$. We say that, $f$ is harmonic in $U$ if $\Delta f = 0$ [6].

From Theorem 3, we obtain

**Corollary 4.** Let

$$\Psi_2(u,v) = (\Psi_2(u,v), \Psi_2(u,v), \Psi_2(u,v), \Psi_2(u,v))$$

be a parametrized surface and assume that $\Psi_2$ is isothermal. Then, $\Psi_2$ is a minimal surface if and only if its coordinate functions $\Psi_2$, $i = 1, 2, 3, 4$ are harmonic.

So, from Corollary 4 we can give the following another corollary:

**Corollary 5.** Let $\Psi_2(u,v)$ be a parametrized isothermal rotation surface which is given by (32). Then it’s coordinate functions are harmonic if and only if $f(u) = c_1 \cos u + c_2 \sin u$ and $g(u) = c_3 \cos u + c_4 \sin u, c_1, c_2, c_3, c_4 \in \mathbb{R}$.

Furthermore, the mean curvature $H$ of the isothermal rotation surface $\Psi_2$ given by (32) is obtained as

$$H = \frac{1}{2} \left( f''g' - f'g'' + f'g' - f''g \right).$$  
(44)

Let the rotation surface $\Psi_2$ given by (32) be isothermal. Then, the Gaussian curvature of the isothermal rotation surface $\Psi_2$ given by (32) is

$$K = \sum_{s=3}^4 \varepsilon_h \varepsilon_s = \frac{1}{2} \left( h_{11}^2 h_{22}^2 - h_{12}^2 h_{21}^2 \right)$$

$$= -(f'g' - f''g)(f'g' - f''g) + (f'g' - f''g)^2.$$  
(45)

**Corollary 6.** The isothermal rotation surface $\Psi_2$ given by (32) is flat if and only if $g(u) = c_1 e^u + c_2 e^{-u}, c_1, c_2 \in \mathbb{R}$.

We call a surface Weingarten surface if the mean curvature $H$ and the Gaussian curvature $K$ satisfy a nontrivial relation $\Phi(H,K) = 0$ [5]. So, we have the following:

**Theorem 4.** The isothermal rotation surface $\Psi_2$ given by (32) is a Weingarten surface which satisfies the condition $HK = 0$ if and only if $f = c_1, c_2 \in \mathbb{R}$.
Proof. From (35), (44) and (45), by a direct computation we get
\[ HK = \frac{1}{2} f'(g^+ - g). \]
If \( HK = 0 \), then \( f' = 0 \). Because, if \( g^+ - g = 0 \), then \( K = 0 \) and so, the relation \( HK = 0 \) is trivial. Hence, we have \( f = c_1, c_1 \in \mathbb{R} \). \( \blacksquare \)

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