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The Estimate for Weakly A-harmonic Tensors with Lipschitz Norm and BMO Norm

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***Abstract* – The Lipschitz norm estimate and BMO norm estimate of weakly A-harmonic tensors are given.**

***Keywords* – Weakly A-Harmonic Tensor, Differential Form, Imbedding Inequality.**

I. Introduction

Let Ω be a connected open subset of R^n, $e_1, e_2, \cdots, e_n$ be the standard unit basis of R^n, and $\wedge^l = \wedge^l(R^n)$ be the linear space of $l-$covectors, spanned by the exterior products $e_I = e_{i1} \wedge e_{i2} \wedge \cdots \wedge e_{il}$, corresponding to all ordered $l-$ tuples $I = (i_1, i_2, \cdots, i_l)$, $1 \le i_1 < i_2 < \cdots < i_l \le n$, $l = 0,1,\cdots,n.$ The Grassman algebra $\wedge = \oplus \wedge^l$ e is a graded algebra with respect to the exterior products. For $\alpha = \sum \alpha^I e_I \in \wedge$ and $\beta = \sum \beta^I e_I \in \wedge$, the inner product in $\wedge$ is given by $<\alpha, \beta> = \sum \alpha^I \beta^I$ with summation over all $l-$ tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0,1,\cdots,n.$ We define the Hodge star operator $*: \wedge \to \wedge$ by the rule $*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$, and $\alpha \wedge *\beta = \beta \wedge *\alpha = <\alpha, \beta> (*1)$ for all $\alpha,\ \beta \in \wedge.$ The norm of $\alpha \in \wedge$ is given by the formula $|\alpha|^2 = <\alpha, \alpha> = *(\alpha \wedge *\alpha) \in \wedge^0 = R.$ The Hodge star is an isometric isomorphism on $\wedge$ with $*: \wedge^l \to \wedge^{n-l}$ and $**(-1)^{l(n-l)}: \wedge^l \to \wedge^l.$ Balls are denoted by B and ρB is the ball with the same center as B and with diam $(\rho B) = \rho \text{diam}(B).$ We do not distinguish balls from cubes throughout this paper. The $n-$ dimensional Lebesgue measure of a set $E \subseteq R^n$ is denoted by $|E|^2$.

Differential forms are important generalizations of real functions and distributions, note that a $0-$ form is the usual function in R^n. A differential $l-$form ω on Ω is a Schwartz distribution on Ω with values in $\wedge^l(R^n).$ We use $D'(\Omega, \wedge^l)$ to denote the space of all differential $l-$forms $\omega(x) = \sum_I \omega_I(x) dx_I = \sum \omega_{i_1 i_2 \cdots i_l}(x) dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$. We write $L^p(\Omega, \wedge^l)$ for the $l-$forms with $\omega_I \in L^p(\Omega, R)$ for all ordered $l-$tuples I. Thus $L^p(\Omega, \wedge^l)$ is a Banach space with norm

$$\|\omega\|_{p,\Omega} = \left(\int_\Omega |\omega(x)|^p dx\right)^{1/p} = \left(\int_\Omega \left(\sum |\omega_I(x)|^2\right)^{p/2} dx\right)^{1/p}.$$

For $\omega \in D'(\Omega, \wedge^l)$ the vector-valued differential form $\nabla \omega = \left(\frac{\partial \omega}{\partial x_1}, \cdots, \frac{\partial \omega}{\partial x_n}\right)$ consists of differential forms $\frac{\partial \omega}{\partial x_i} \in D'(\Omega, \wedge^l)$ where the partial differentials are applied to the coefficients of $\omega.$ As usual, $W^{1,p}(\Omega, \wedge^l)$ is used to denote the Sobolev space of $l-$ forms, which equals $L^p(\Omega, \wedge^l) \cap L_1^p(\Omega, \wedge^l)$ with norm
$\|\omega\|_{W^{1,p}(\Omega, \wedge^l)} = \text{diam}(\Omega)^{-1} \|\omega\|_{p,\Omega} + \|\nabla \omega\|_{p,\Omega}$.
The notations $W_{loc}^{1,p}(\Omega, R)$ and $W_{loc}^{1,p}(\Omega, \wedge^l)$ are self - explanatory. We denote the exterior derivative by $d: D'(\Omega, \wedge^l) \to D'(\Omega, \wedge^{l+1})$ for $l = 0,1,\cdots,n.$ Its formal adjoint operator $d^*: D'(\Omega, \wedge^{l+1}) \to D'(\Omega, \wedge^l)$ is given by $d^* = (-1)^{nl+1} * d *$ on $D'(\Omega, \wedge^{l+1})$, $l = 0,1,\cdots,n.$

Let $u \in L_{loc}^1(\Omega, \wedge^l)$, $l = 0,1,\cdots,n.$ We write $u \in locLip_k(\Omega, \wedge^l)$, $0 \le k \le 1,$ if

$$\|u\|_{locLip_k,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-(n+k)/n} \|u - u_Q\|_{1,Q} < \infty \tag{1.1}$$

for some $\sigma \ge 1$. Further, we write $Lip_k(\Omega, \wedge^l)$, for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|u\|_{Lip_k,\Omega}$ for this norm. Similarly, for $u \in L_{loc}^1(\Omega, \wedge^l)$, $l = 0,1,\cdots,n$, we write $u \in BMO(\Omega, \wedge^l)$ if

$$\|u\|_{*,\Omega} = \sup_{\sigma Q \subset \Omega} |Q|^{-1} \|u - u_Q\|_{1,Q} < \infty \tag{1.2}$$

for some $\sigma \ge 1$. When u is a $0-$form, (1.2) reduces to the classical definition of $BMO(\Omega)$.

From [1, 18], if $D \subset R^n$ be a bounded, convex domain, to each $y \in D$ there corresponds a linear operator $K_y: C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ defined by

$$(K_y \omega)(x; \xi_1, \cdots, \xi_{l-1}) = \int_0^1 t^{l-1} \omega(tx + y - ty; x - y, \xi_1, \cdots, \xi_{l-1}) dt$$

and a decomposition $\omega = d(K_y \omega) + K_y(d\omega).$

A homotopy operator $T: C^\infty(D, \wedge^l) \to C^\infty(D, \wedge^{l-1})$ is defined by averaging K_y over all points y in D, i.e.,

$$T\omega = \int_D \varphi(y) K_y \omega dy, \tag{1.3}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y) dy = 1.$ Then, there is also a decomposition

$$\omega = d(T\omega) + T(d\omega). \tag{1.4}$$

The $l-$form $\omega_D \in D'(D, \wedge^l)$ is defined by

$$\omega_D = \begin{cases} |D|^{-1} \int_D \omega(y) dy & if\ l = 0 \\ d(T\omega) & if\ l = 1,2,\cdots,n \end{cases}$$

for all $\omega \in L^p(D, \wedge^l), 1 \le p < \infty.$ Then $\omega_D = \omega - T(d\omega).$ Clearly ω_D is a closed form and for $l > 0$, ω_D is an exact form.

By substituting $z = tx + y - ty$, (1.3) reduces to

$$T\omega(x, \xi) = \int_D \omega(z, \zeta(z, x - z, \xi)) dz, \tag{1.5}$$

Where the vector function $\zeta: D \times R^n \to R^n$ is given by

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$\zeta(z,h)=h\int_0^\infty s^{l-1}(1+s)^{n-1}\varphi(z-sh)ds.$

Integral (1.5) defines a bounded operator

$T:L^s(D,\wedge^l)\to W^{1,s}(D,\wedge^{l-1}),\ l=1,2,\cdots,n.$

with norm estimated by

$\|Tu\|_{W^{1,s}(D)}\le C|D|\|u\|_{s,D}$.

Given $g\in L^r(\Omega,\wedge^l)$ and $f\in L^{r/(p-1)}(\Omega,\wedge^l)$ where $r\ge\max\{1,p-1\}$, we consider the non homogeneous equation for differential forms

$$d^*A(x,g+du)=d^*f \text{ for } u\in W^{1,r}_{loc}(\Omega,\wedge^l) \tag{1.6}$$

Where $A:\Omega\times\wedge^l(R^n)\to\wedge^{l+1}(R^n)$ satisfied the conditions

(H1) $|A(x,\xi)-A(x,\zeta)|\le\beta|\xi-\zeta|(|\xi|+|\zeta|)^{p-2}$,

(H2) $<A(x,\xi)-A(x,\zeta),\ \xi-\zeta>\ge\alpha|\xi-\zeta|^2(|\xi|+|\zeta|)^{p-2}$,

(H3) $A(x,\lambda\xi)=|\lambda|^{p-2}\lambda A(x,\zeta)$, (1.7)

For almost every $x\in\Omega,\lambda\in R$ and all $\xi,\zeta\in\wedge^l(R^n)$. Here $\alpha,\beta>0$ are constants and $1<p<\infty$ is a fixed exponent associated with (1.6).

When $g=0$ and $d^*f=0$, equation (1.6) becomes

$$d^*A(x,du)=0. \tag{1.8}$$

There has been remarkable work (see [1-10]) in the study of equation (1.8) . When u is a $0-$form, that is , u is a function, (1.8) is equivalent to

$$\text{div}A(x,\nabla u)=0. \tag{1.9}$$

Lots of results have been obtained in recent years about different versions of the A-harmnic equation, see[11-15].

In 1995, B. Stroffolini [16] first introduced weakly A-harmonic tensors. The word *weak* means that the integrable exponent r of u is smaller than the natural exponent p.

Definition 1.1 [16] A very weak solution to (1.6) (also called weakly A-harmonic tensor) is an element u of the Sobolev space $W^{1,r}_{loc}(\Omega,\wedge^{l-1})$ with $\max\{1,p-1\}\le r<p$ such that

$$\int_\Omega\langle A(x,g+du),d\varphi\rangle dx=\int_\Omega\langle f,d\varphi\rangle dx \tag{1.10}$$

for all $\varphi\in W^{1,\frac{r}{r-p+1}}(\Omega,\wedge^{l-1})$ with compact support.

In this paper, we continue to consider the weakly A-harmonic tensor. In many situations, the process to study solutions of PDE involves estimating the various norms of the operators. Hence we ar e motivated to establish some Lipschitz norm inequalities and BMO norm inequalities for homotopy operator to the versions of differential forms. Based on the weak reverse Holder inequality of weakly A-harmonic tensor, we establish the imbedding inequalities and poincare inequality of weakly A-harmonic tensor, then given the Lipschitz norm estimate and BMO norm estimate of weakly A-harmonic tensor.

II. Notion and Lemmas

From results appearing in [18], we have the following lemma.

Lemma 2.1 Let $u\in L^s_{loc}(B,\wedge^l),l=1,2,\cdots,n,1<s<\infty$, be a differential form in a ball $B\subset R^n$.

$\|\nabla(Tu)\|_{s,B}\le C|B|\|u\|_{s,B}$,

$\|Tu\|_{s,B}\le C|B|\text{diam(B)}\|u\|_{s,B}$.

We will need the following generalized Holder inequality.

Lemma 2.2 Let $0<\alpha<\infty,0<\beta<\infty$ and $s^{-1}=\alpha^{-1}+\beta^{-1}$. If f and g are measurable functions on R^n, then

$\|fg\|_{s,\Omega}\le\|f\|_{\alpha,\Omega}\cdot\|g\|_{\beta,\Omega}$

For any $\Omega\subset R^n$.

We need the following weak reverse Holder inequality of weakly A-harmonic tensors.

Lemma 2.3 Given the A-harmnic equation (1.6), let $\varepsilon=\varepsilon(n,p,\alpha,\beta)\in(0,p-1)$. Suppose that $u\in W^{1,r_1}(\Omega,\wedge^{l-1})$ is an weakly A-harmonic tensor for some $r_1\in(p-\varepsilon,p)$. Then for any concentric cubes $Q\subset 2Q\subset\Omega$, ,we have

$$\left(\int_Q|du|^{r_1}\right)^{1/r_1}\le C(n,p)\left(\int_{2Q}|du|^{r_2}\right)^{1/r_2}$$

Where

$$r_2=\max\left\{\frac{nr_1}{n+r_1-1},\frac{nr_1}{np-n+r_1-p+1}\right\}.$$

Here $r_2<r_1,1<p<\infty$, the constant $C(n,p)$ does not depend on r_1 and r_2..

III. Inequalities of Weakly A-Harmonic Tensor

Now, we prove the following imbeding inequality of weakly A-harmonic tensor .

Theorem 3.1 Let $u\in W^{1,r_1}_{loc}(\Omega,\wedge^{l-1}),\quad l=1,2,\cdots,n,$ $\max\{1,p-1\}\le r<p,$ be a weakly A-harmonic tensor satisfying (1.6) in a bounded domain $\Omega\subset R^n$ and $T:C^\infty(\Omega,\wedge^l)\to C^\infty(\Omega,\wedge^{l-1})$ be a homotopy operator. Then there exists a constant C independent of u such that

$$\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}\le C(n,p)|B|\left(\int_{2B}|du|^r\right)^{1/r}, \tag{3.1}$$

$$\left(\int_B|T(du)|^r dx\right)^{1/r}\le C(n,p)|B|\left(\int_{2B}|du|^s\right)^{1/s}, \tag{3.2}$$

for all balls B with $2B\subset\Omega$, where $s<r$,

$$s=\max\left\{\frac{nr}{n+r-1},\frac{nr}{np-n+r-p+1}\right\}. \tag{3.3}$$

Proof Let $u\in W^{1,r_1}_{loc}(\Omega,\wedge^{l-1}),\quad l=1,2,\cdots,n,$ be a very weak solution of (1.6). By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}&=\|\nabla(T(du))\|_{r,B}\\ &\le C|B|\|du\|_{r,B}\\ &=C|B|\left(\int_B|du|^r dx\right)^{1/r}\\ &\le C(n,p)|B||B|^{1/r}\left(\int_{2B}|du|^s\right)^{1/s},\end{aligned} \tag{3.4}$$

where s is as in (3.3). Note that (3.4) can be written as

$$\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}\le C(n,p)|B|\left(\int_{2B}|du|^r\right)^{1/r}. \tag{3.5}$$

For $s<r$, by (3.4) and Lemma2.2, we have

$$\begin{aligned}&\left(\int_B|\nabla(T(du))|^r dx\right)^{1/r}\\ &\le C(n,p)|B||B|^{1/r}|B|^{-1/s}\left(\int_{2B}|du|^s\right)^{1/s}\\ &\le C(n,p)|B||B|^{1/r}|B|^{-1/s}|\text{B}|^{\frac{r-s}{rs}}\left(\int_{2B}|du|^r\right)^{1/r}\\ &=C(n,p)|B|\left(\int_{2B}|du|^r\right)^{1/r}.\end{aligned} \tag{3.6}$$

Now we prove the following Poincare-type inequality for $T(u)$ with the L^s – norm which plays an important role in this paper.

Theorem 3.2 Let $u \in W^{1,r_1}_{loc}(\Omega, \wedge^{l-1})$, $l = 1, 2, \cdots, n$, $\max\{1, p-1\} \le r < p$, be a weakly A-harmonic tensor satisfying (1.6) in a bounded domain $\Omega \subset R^n$ and $T: C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator. Then, there exists a constant C independent of u such that

$$(\int_B |Tu - (Tu)_B|^r\, dx)^{1/r} \le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|u|^s dx)^{1/s}, \quad (3.7)$$

$$(\int_B |Tu - (Tu)_B|^r\, dx)^{1/r} \le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|u|^r dx)^{1/r}, \quad (3.8)$$

for all balls B with $2B \subset \Omega$, where s is as in (3.3), $s < r$.

Proof: Let $u \in W^{1,r_1}_{loc}(\Omega, \wedge^{l-1})$ be a very weak solution of (1.6). By Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}(\int_B |T(du)|^r\, dx)^{1/r} &= \|T(du)\|_{r,B} \\ &\le C|B|\mathrm{diam}(B)\|du\|_{r,B} \\ &= C|B|\mathrm{diam}(B)(\int_B |du|^r\, dx)^{1/r} \\ &\le C(n,p)|B|\,diam(B)|B|^{1/r}(\int_{2B}|du|^s)^{1/s},\end{aligned} \quad (3.9)$$

where s is as in (3.3). Note that (3.9) can be written as

$$(\int_B |T(du)|^r\, dx)^{1/r} \le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|du|^s)^{1/s}. \quad (3.10)$$

For $Tu - (Tu)_B = Td(Tu)$, we have

$$\begin{aligned}&(\int_B |Tu - (Tu)_B|^r\, dx)^{1/r} \\ &= (\int_B |Td(Tu)|^r\, dx)^{1/r} \\ &\le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|d(Tu)|^s dx)^{1/s}\end{aligned} \quad (3.11)$$

for all balls B with $2B \subset \Omega$, where s is as in (3.3).

We all know that for any differential form u, $d(T(u)) = u_B$ and $\|u_B\|_{p,B} \le C\|u\|_{p,B}$. Hence

$$\begin{aligned}&(\int_B |Tu - (Tu)_B|^r\, dx)^{1/r} \\ &\le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|d(Tu)|^s dx)^{1/s} \\ &= C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|u_B|^s dx)^{1/s} \\ &\le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|u|^s dx)^{1/s}.\end{aligned}$$

Using Holder inequality, we have

$$(\int_B |Tu - (Tu)_B|^r\, dx)^{1/r} \le C(n,p)|B|\,\mathrm{diam}(B)(\int_{2B}|u|^r dx)^{1/r}.$$

IV. The Estimate for Homotopy Operators with Lipschitz Norm and BMO Norm

Now we will establish the following estimate with Lipschitz norm and BMO norm.

Theorem 4.1 Let $u \in W^{1,r_1}_{loc}(\Omega, \wedge^{l-1})$, $l = 1, 2, \cdots, n$, $\max\{1, p-1\} \le r < p$, be a weakly A-harmonic tensor satisfying (1.6) in a bounded domain $\Omega \subset R^n$ and $T: C^\infty(\Omega, \wedge^l) \to C^\infty(\Omega, \wedge^{l-1})$ be a homotopy operator. Then, there exists a constant C independent of u such that

$$\|T(u)\|_{*,\Omega} \le \|T(u)\|_{locLip_k,\Omega} \le C\|u\|_{s,\Omega}. \quad (4.1)$$

where s is as in (3.3), s < r.

Proof: By the definition of the BMO norm, we have

$$\begin{aligned}&\|T(u)\|_{*,\Omega} \\ &= \sup_{\sigma B\subset\Omega}(\mu(B))^{-1}\|T(u) - (T(u))_B\|_{1,B} \\ &= \sup_{\sigma B\subset\Omega}(\mu(B))^{k/n}(\mu(B))^{-(n+k)/n}\|T(u) - (T(u))_B\|_{1,B} \\ &\le C\sup_{\sigma B\subset\Omega}(\mu(B))^{-(n+k)/n}\|T(u) - (T(u))_B\|_{1,B} \\ &\le C\|T(u)\|_{locLip_k,\Omega}.\end{aligned}$$

By the definition of the Lipschitz norm, Holder's inequality with $1 = 1/r + (r-1)/r$ and (3.7), we have

$$\begin{aligned}&\|T(u)\|_{locLip_k,\Omega} \\ &= \sup_{\sigma B\subset\Omega}(\mu(B))^{-(n+k)/n}\|T(u) - (T(u))_B\|_{1,B} \\ &\le \sup_{\sigma B\subset\Omega}(\mu(B))^{-(n+k)/n}(\int_B |T(u) - (T(u))_B|^r dx)^{1/r}(\int_B 1^{r/(r-1)} dx)^{(r-1)/r} \\ &= \sup_{\sigma B\subset\Omega}(\mu(B))^{-(n+k)/n+(r-1)/r}\|T(u) - (T(u))_B\|_{r,B} \\ &\le \sup_{\sigma B\subset\Omega}(\mu(B))^{-(n+k)/n+(r-1)/r}C(n,p)|B|\,\mathrm{diam}(B)|B|^{-(\frac{1}{s}-\frac{1}{r})}\|u\|_{s,2B} \\ &\le C(n,p)|\Omega|^{-(n+k)/n+(r-1)/r+1+1/n-1/s+1/r}\|u\|_{s,\Omega} \\ &\le C(n,p)\|u\|_{s,\Omega},\end{aligned}$$

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