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Convolved k-Fibonacci Sequences

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Abstract **– In this paper we study the iterated convolutionof the k-Fibonacci sequences. A particular case is for the self-convolution of the sequence $F_{k,n}$ $(n \in N\}$. Besides the generating functions of all these convolved sequences, we find the recurrence relation between the terms of the resulting sequences.**

Keywords **– Fibonacci numbers, k-Fibonacci Sequences, Convolution.**

I. Introduction

In [6], the convolved Fibonacci sequences are defined in the form $F_n^{(r)} = \sum_{j=0}^{n} F_j F_{n-j}^{(r-1)}$ with initial condition $F_n^{(0)} = F_n$ where F_n are the classical Fibonacci numbers.

The aim of this paper consists of extending this concept to case of the k-Fibonacci numbers.

1.1. *Definition 1.*

For any integer number k≥1, the k–Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$ is defined recurrently by $F_{k,n+1} = k\ F_{k,n} +$ $F_{k,n-1}$ with initial conditions $F_{k,0} = 0,\ \ F_{k,1} = 1.$

Characteristic equation from the definition is $r^2 = k\ r + 1$ whose solutions are $\sigma_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \dfrac{k-\sqrt{k^2+4}}{2}$ that verify $\sigma_1 \cdot \sigma_2 = -1,\ \sigma_1 + \sigma_2 = k,\ \ \sigma_1 - \sigma_2 = \sqrt{k^2+4},\ \ \sigma^2 = k\sigma + 1,\ \ \sigma_1 > 0,\ \ \sigma_2 < 0.$

For the properties of the k–Fibonacci numbers, see [3,4]. In particular, the Binet Identity is,

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \tag{1}$$

From the Binet Identity it is easy to prove (see [4] Formula (10)) the formula for find the k-Fibonacci numbers.

$$F_{k,n} = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j}{j} k^{n-2j} \tag{2}$$

Generating function of the k–Fibonacci numbers is $f(x) = \dfrac{x}{1-k\,x-x^2}$

Finally, negative k – Fibonacci numbers are defined $F_{k,-n} = (-1)^{n+1} F_{k,n}$

1.2. *Definition 2.*

For any integer number k≥1, the k–Lucas sequence, say $\{L_{k,n}\}_{n\in N}$, is defined recurrently by [1] $L_{k,n+1} = k\,L_{k,n}$

$+L_{k,n-1}$ with initial conditions $L_{k,0} = 2, \ L_{k,1} = k.$

The Binet Identity for the k–Lucas numbers is $\{L_{k,n}\}_{n\in N}$.

The k–Lucas numbers are related to the k–Fibonacci numbers by the relation $L_{k,n} = F_{k,n-1} + F_{k,n+1}$. From this relation, it is easy to prove that $F_{k,n} = \dfrac{L_{k,n-1} + L_{k,n+1}}{k^2+4}$.

Generating function of the k–Lucas numbers is $l(x) = \dfrac{2-kx}{1-kx-x^2}$. Moreover, $L_{k,-n} = (-1)^n L_{k,n}$.

II. Convolved k - Fibonacci Sequences

A convolved k–Fibonacci sequence is obtained applying a convolution operation to the k–Fibonacci sequence one or more times. Specifically, define $F_{k,n}^{(0)} = F_{k,n}$ and $F_{k,n}^{(r)} = \sum_{j=0}^{n} F_{k,j} F_{k,n}^{(r-1)}$.

From the definition.

$$F_k^{(0)} = \{F_{k,n}^{(0)}\} = \{F_{k,n}\} = \{0,1,k,k^2+1,k^3+2k,k^4+3k^2+1,k^5+4k^3+3k,\ldots\}$$

$$F_k^{(1)} = \{F_{k,n}^{(1)}\} = \{0,0,1,2k,3k^2+2,4k^3+6k,5k^4+12k^2+3,\ldots\}$$

$$F_k^{(2)} = \{F_{k,n}^{(2)}\} = \{0,0,0,1,3k,6k^2+3,10k^3+12k,15k^4+30k^2+6,\ldots\}$$

$$F_k^{(3)} = \{F_{k,n}^{(3)}\} = \{0,0,0,0,1,4k,10k^2+4,20k^3+20k,35k^4+60k^2+10,\ldots\}$$

$$F_k^{(4)} = \{F_{k,n}^{(4)}\} = \{0,0,0,0,0,1,5k,15k^2+5,35k^3+30k,70k^4+105k^2+15,\ldots\}$$

….

In particular, for k = 1 and the classical Fibonacci numbers, $F^{(r)} = \{F_n^{(r)}\}$ and

$$F^{(0)} = \{0,1,1,2,3,5,8,13,21,34,55,\ldots\}$$

$$F^{(1)} = \{0,0,1,2,5,10,20,38,71,130,235,\ldots\}$$

$$F^{(2)} = \{0,0,0,1,3,9,22,51,111,233,474,942,\ldots\}$$

$$F^{(3)} = \{0,0,0,0,1,4,14,40,105,256,594,1324,2860,\ldots\}$$

$$F^{(4)} = \{0,0,0,0,0,1,5,20,65,190,511,1295,3130,7285,\ldots\}$$

…..

All these sequences are indexed in the OEIS [5].

For k = 2 and the Pell numbers $F_{2,n}^{(r)} = P_n^{(r)}$:

$$P^{(0)} = \{0,1,2,5,12,29,70,169,408,985,\ldots\}$$

$P^{(1)} = \{0, 0, 1, 4, 14, 44, 131, 376, 1052, 2888, \ldots\}$

$P^{(2)} = \{0, 0, 0, 1, 6, 27, 104, 366, 1212, 3842, 11784, \ldots\}$

$P^{(3)} = \{0, 0, 0, 0, 1, 8, 44, 200, 810, 3032, 10716, \ldots\}$

$P^{(4)} = \{0, 0, 0, 0, 0, 1, 10, 65, 340, 1555, 6482, \ldots\}$

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All these last sequences are cited in the OEIS but without the initial null terms.

A last example: for k = 3:

$F_{3,n}^{(0)} = \{0, 1, 3, 10, 33, 109, 360, 1189, 3927, 12970, \ldots\}$

$F_{3,n}^{(1)} = \{0, 0, 1, 6, 29, 126, 516, 2034, 7807, 29382, 108923, \ldots\}$

$F_{3,n}^{(2)} = \{0,0,0,1,9,57,306,1491,6813,29737,125406,\ldots\}$

$F_{3,n}^{(3)} = \{0,0,0,0,1,12,94,600,3385,17568,85826,\ldots\}$

$F_{3,n}^{(4)} = \{0,0,0,0,0,1,15,140,1035,6630,38493,\ldots\}$

For k ≥ 3, the only sequences indexed in the OEIS are that for r = 0.

2.1. *Formula for the General Term of the Convolved K-Fibonacci Sequence*

By induction, we can prove the following identities for the elements $F_{k,n}^{(r)}$ of the convolved k–Fibonacci sequences:

$$F_{k,n}^{(r)} = 0 \text{ for } 0 \le n \le r \text{ and } F_{k,r+p+1}^{(r)} = \sum_{j=0}^{p} \binom{p-j}{j} \binom{r+p-j}{p-j} k^{p-2j}$$

If r = 0, this formula becomes the Formula (1) to obtain the k-Fibonacci numbers.

2.2. *Theorem*

Convolved k-Fibonacci sequences verify the recurrence relation.

$$F_{k,n+1}^{(r)} = k\,F_{k,n}^{(r)} + F_{k,n-1}^{(r)} + F_{k,n}^{(r-1)} \text{ for n, r} \ge 1. \tag{3}$$

Proof.

We will prove it by induction.

$$F_{k,n+1}^{(1)} = \sum_{j=0}^{n+1} F_{k,j} F_{k,n+1-j} = \sum_{j=0}^{n+1} F_{k,j} \left(k\,F_{k,n-j} + F_{k,n-1-j} \right)$$

$$= k \sum_{j=0}^{n} F_{k,j} F_{k,n-j} + \sum_{j=0}^{n-1} F_{k,j} F_{k,n-1-j} + k\,F_{k,n+1} F_{k,-1} + F_{k,n} F_{k,n-1} + F_{k,n+1} F_{k,-2} =$$

$$= k\, F_{k,n}^{(1)} + F_{k,n-1}^{(r)} + F_{k,n}^{(0)}$$

Let us suppose the formula is true until r ≥ 2. $F_{k,n+1}^{(r)} = k\, F_{k,n}^{(r)} + F_{k,n-1}^{(r)} + F_{k,n}^{(r-1)}$. Then

$$F_{k,n+1}^{(r+1)} = \sum_{j=0}^{n+1} F_{k,j} F_{k,n+1-j}^{(r)} = \sum_{j=0}^{n+1} F_{k,j} \left(k\, F_{k,n-j}^{(r)} + F_{k,n-1-j}^{(r)} + F_{k,n-j}^{(r-1)} \right) =$$

$$= \sum_{j=0}^{n} F_{k,j} \left(k\, F_{k,n-j}^{(r)} + F_{k,n-1-j}^{(r)} + F_{k,n-j}^{(r-1)} \right) + F_{k,n+1} F_{k,0}^{(r)} \quad (\text{because } F_{k,0}^{(r)} = 0)$$

$$= k \sum_{j=0}^{n} F_{k,j} F_{k,n-j}^{(r)} + \sum_{j=}^{n-1} F_{k.j} F_{k,n-1-j}^{(r)} + \sum_{j=0}^{n} F_{k,j} F_{k,n-j}^{(r-1)} + F_{k,n} F_{k,-1}^{(r)} \quad (\text{because } F_{k,1}^{(r)} = 0)$$

$$= k\, F_{k,n}^{(r+1)} + F_{k,n-1}^{(r+1)} + F_{k,n}^{(r)}$$

2.3. *Convolved k-Fibonacci Sequences and the Fibonacci Polynomials*

The sequences $F_k^{(r)}$ are related to the Fibonacci polynomials by the relation $F_{k,n}^{(r)} = \frac{1}{r\,!} \frac{d^r F_{k,n}}{dk^r}$ where $\frac{d^r F_{k,n}}{dk^r}$ is the derivative of order r with respect to k of the k–Fibonacci numbers of Definition.

2.4. *Generating Function*

In [7], Formula (2.2.3), the following theorem is proven:

If f(x) and g(x) are the respective generating functions of the sequences {u_n ç and {v_n}, then f(x)· g(x) is the generating function of the convolution of these sequences.

So, and taking into account the generating function of the k–Fibonacci numbers is $f(x) = \frac{x}{1 - k\,x - x^2}$, the generating function of the convolved k-Fibonacci sequences is,

$$f^{(r)}(x) = \left(\frac{x}{1 - k\,x - x^2} \right)^{r+1} \tag{4}$$

As a special case, $F_{k,n}^{(1)}$ is the self-convolution of the $F_{k,\,n,}$ numbers.

Sometimes, the convolution of the sequences U = {u_n}, V = {v_n} is represented as \$U ⊗ V = {u_n ⊗ v_n}\$, so the self-convolution $F_{k,n}^{(1)} = \left\{ F_{k,n} \otimes F_{k,n} \right\}$

2.5. *Theorem*

The self-convolution of the k–Fibonacci numbers verifies the formula,

$$F_{k,n}^{(1)} = \sum_{j=0}^{n} F_{k,j} F_{k,n-j} = \frac{n\, L_{k,n} - k\, F_{k,n}}{k^2 + 4} \tag{5}$$

Proof.

Applying the Binet Identity (1):

$$F_{k,n}^{(1)}=\sum_{j=0}^{n}F_{k,j}F_{k,n-j}=\frac{1}{k^2+4}\sum_{j=0}^{n}\left(\sigma_1^j-\sigma_2^j\right)\left(\sigma_1^{n-j}-\sigma_2^{n-j}\right)=$$

$$=\frac{1}{k^2+4}\sum_{j=0}^{n}\left(\sigma_1^n+\sigma_2^n-\sigma_1^n\left(\frac{\sigma_2}{\sigma_1}\right)^j-\sigma_2^n\left(\frac{\sigma_1}{\sigma_2}\right)^j\right)=$$

$$=\frac{1}{k^2+4}\left((n+1)L_{k,n}-\sigma_1^n\frac{\left(\frac{\sigma_2}{\sigma_1}\right)^{n+1}-1}{\frac{\sigma_2}{\sigma_1}-1}-\sigma_2^n\frac{\left(\frac{\sigma_1}{\sigma_2}\right)^{n+1}-1}{\frac{\sigma_1}{\sigma_2}-1}\right)=$$

$$=\frac{1}{k^2+4}\left((n+1)L_{k,n}-\frac{\sigma_2^{n+1}-\sigma_1^{n+1}}{\sigma_2-\sigma_1}-\frac{\sigma_1^{n+1}-\sigma_2^{n+1}}{\sigma_1-\sigma_2}\right)=\frac{1}{k^2+4}\left((n+1)L_{k,n}-2F_{k,n+1}\right)=$$

$=\frac{1}{k^2+4}\left(nL_{k,n}+F_{k,n+1}+F_{k,n-1}-2F_{k,n+1}\right)=\frac{1}{k^2+4}\left(nL_{k,n}-(F_{k,n+1}-F_{k,n-1})\right)=\frac{nL_{k,n}-kF_{k,n}}{k^2+4}$ For instance, for the Pell numbers $F_{2,n}=P_n$ it is $P_n^{(1)}=\frac{(n-1)P_n+nP_{n-1}}{4}$

2.6. *Theorem*

Self-convolution of the k–Fibonacci numbers verifies the recurrence relation $F_{k,n+1}^{(1)}=2k\,F_{k,n}^{(1)}-(k^2-2)$ $F_{k,n-1}^{(1)}-2k\,F_{k,n-2}^{(1)}-F_{k,n-3}^{(1)}$

Proof.

Taking into account that $L_{k,n}=F_{k,n+1}+F_{k,n-1}$ the formula (5) can be expressed as $F_{k,n}^{(1)}=\sum_{j=0}^{n}F_{k,j}F_{k,n-j}$ $=\frac{k(n-1)F_{k,n}+2n\,F_{k,n-1}}{k^2+4}$ and so, $F_{k,n}^{(1)}=\frac{(k^2n+2n+2)F_{k,n}+k\,n\,F_{k,n-1}}{k^2+4}$

From the definition of the k–Fibonacci numbers, the following relations are found:

Coefficients of $F_{k,n}$:

$$(k^2+4)F_{k,n}^{(1)}:2k(k\,n-k)=2k^2n-2k^2$$

$$(k^2+4)F_{k,n-1}^{(1)}:-(k^2-2)(2n-2)=-2k^2n+2k^2+4n-4$$

$$(k^2+4)F_{k,n-2}^{(1)}:-2k(-k\,n+k)=2k^2n-2k^2$$

$$(k^2+4)F_{k,n-3}^{(1)}:-(k^2n+2n-2k^2-6)=-k^2n+2k^2-2n+6$$

$$(k^2+4)F_{k,n+1}^{(1)}:k\,n^2+2n+2$$

And adding the for first row the fifth row is found. Coefficients of $F_{k,n-1}$:

$$(k^2+4)F_{k,n}^{(1)}:2k\,2n-4k\,n$$

$$(k^2+4)F_{k,n-1}^{(1)}:(-k^2+2)(-k\,n)=k^3n-2k\,n$$

$$(k^2+4)F^{(1)}_{k,n-2} :-2k(k^2n+2n-k^2-4)=-2k^3n-4kn+2k^3+8k$$

$$(k^2+4)F^{(1)}_{k,n-3} :-(-k^3n-3kn+2k^3+8k)=k^3n+3kn-2k^3-8k$$

$$(k^2+4)F^{(1)}_{k,n+1} :kn$$

Again, adding the for first row the fifth row is found.

Obviously, this method is more and more laborious the higher the value of “r”. But there is another way to find these recurrences if we take into account that the denominator of the generating function corresponds to the recurrence relation between the terms of the sequence. The expansion of the respective

denominators leads us to the recurrence relation of the corresponding sequence without more than changing x^p by $F^{(r)}_{k,n-p}$. So, from Equation (4) and taking into account that $1=x^0=F^{(0)}_{k,n}$ it is

$$r=0\to 1-kx-x^2=0\to 1=kx+x^2\to F^{(0)}_{k,n}=kF^{(0)}_{k,n-1}+F^{(0)}_{k,n-2}\ \ ((F_{k,n+1}=kF_{k,n}+F_{k,n-1}))$$

$$r=1\to(1-kx-x^2)^2=0\to 1=2k-(k^2-2)x^2-2kx^3-x^4\to\cdots$$

$$\to F^{(1)}_{k,n}=2kF^{(1)}_{k,n-1}-(k^2-2)F^{(1)}_{k,n-2}-2kF^{(1)}_{k,n-3}-F^{(1)}_{k,n-4}$$

$$r=2\to(1-kx-x^2)^3=0\to 1=3kx+(3-3k^2)x^2-(6k-k^3)x^3-(3-3k^2)x^4+3kx^5-x^6\to$$
$$\to F^{(2)}_{k,n}=3kF^{(2)}_{k,n-1}+(3-3k^2)F^{(2)}_{k,n-2}-(6k-k^3)F^{(2)}_{k,n-3}-(3-3k^2)F^{(2)}_{k,n-4}+3kF^{(2)}_{k,n-5}-F^{(2)}_{k,n-6}$$

2(r + 1) initial conditions are necessary for these relations.

2.7. *Corollary*

For the classical Fibonacci numbers (k = 1), the respective recurrence relations are:

$$F^{(0)}_n=F_n=F_{n-1}+F_{n-2}$$

$$F^{(1)}_n=F_n\otimes F_n=2F^{(1)}_{n-1}+F^{(1)}_{n-2}-F^{(1)}_{n-3}-F^{(1)}_{n-4}$$

$$F^{(2)}_n=F_n\otimes F^{(1)}_n=3F^{(2)}_{n-1}-5F^{(2)}_{n-3}+3F^{(2)}_{n-5}+F^{(2)}_{n-6}$$

$$F^{(3)}_n=F_n\otimes F^{(2)}_n=4F^{(3)}_{n-1}-2F^{(3)}_{n-2}-8F^{(3)}_{n-3}+5F^{(3)}_{n-4}+8F^{(3)}_{n-5}-2F^{(3)}_{n-6}-4F^{(3)}_{n-7}-F^{(3)}_{n-8}$$

III. Conclusions

In this paper, the self-convolution of the k-Fibonacci sequence is first studied.

From there, the process is repeated an iterated number of times and the recurrence relations of the elements that form them are found. These relations are intended to calculate any term of the sequences based on the previous ones.

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