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A Study of W_4 - Symmetric K-Contact Riemannian Manifold

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Abstract: **In this paper the curcature and its characterizations in a K- contact Riemannian manifold are studied. We consider cases when the manifold is W4 - Felt as well as W4 - Symmetric**

Keywords: **W_4, Symmetric, K-Contact, Manifold**

I. Preliminaries

Let M_N be an (n=2m+1) dimensional contact Riemannian manifold with the structure tensors (Φ, ξ, η, g).

Then the following formulas holds:

(1.1) $\phi^2 X = -X + \eta(X)\xi,\ \eta(\xi) = 1$
$\phi\xi = 0,$
$g(X,\xi) = \eta(X),\ \ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$

(1.2) $F(X,Y) = -g(\emptyset X, Y) = +g(X, \emptyset Y) = (\nabla_X \eta)(Y) = -(\nabla_Y \eta)(X)$

(1.3) $d\eta(X,Y) = g(X, \phi Y)$ [1]] for any vector fields X and Y in M.

If we defined an operator h by $\bar{h} = \frac{1}{2} L_\xi \phi$ where h is the lie derivative, then any contact Riemannian manifold satisfies the condition that h and $\phi\bar{h}$ are symmetric operators, h ant-commutes with ϕ (i.e $\phi\bar{h} + \bar{h}\phi$), $\eta o \bar{h} = 0$ (see[2] and [3])

(1.4) $\bar{\mathrm{h}}\xi = 0$ and e ($\nabla_X \xi = -\phi X - \phi\bar{h}$ X), A contact Riemannian manifold is said to be
K-contact if

(1.5) If $\nabla_X \xi = -\phi X$ also in K-contact we have

(1.7) $(\nabla_Y F)(Z,X) = R(Z,X,Y,\xi)$

(1.8) $(\nabla_Z F)(\phi X, \phi Y) + (\nabla_Z F)(X,Y) - \eta(Y)\eta(\nabla_Z \phi X) + \eta(X)\eta(\nabla_Z \phi Y) = 0,$

(1.9) $R(X,Y,Z,\xi) + R(\phi X, \phi Y, \phi Z, \phi\xi) = \eta(Y)\eta(\nabla_Z \phi X - \eta(X)\eta(\nabla_Z \phi Y)),$

(1.10) $\eta(\nabla_Y \emptyset X) = \eta(X)\eta(Y) - g(X,Y)$

(1.11) $(\xi,\xi) = Ric(\xi,\xi) = \eta - 1,$ where R is the Riemannian (0,4) curvature tensors S = Ric (.,.) is the Ricci tensor and $F(X,Y) = g(\phi X, Y)$.

II. $\mathbf{W_4}$ –Tensor in K-Contact Riemannian

Mishra and Pokhariyal [4] gave the definition of W_4-tensor as

(2.1) $W_4(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)\phi Y - g(X,Y)\phi Z]$ where Q is the symmetric endomorphism of tangent space a point to the Ricci tensor or

(2.2) $W_4{}'(X,Y,Z,U) = R(X,Y,Z)U + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,Y)Ric(Z,U)]$

Definition (2.1): A $K-contact$ Riemannian manifold is said to be $W_4 - flat$ if $W_4{}'(X,Y,Z,U) = 0$

Theorem 2.1: The W_4 curvature tensor in a K-contact Riemannian manifold satisfies the following properties.

2.1(a). $W_4{}'(X,Y,Z,\xi) = \eta(x)g(Y,Z) - \eta(Z)g(X,Y)$

2.1(b).$W_4{}'(\xi,Y,Z,\xi) = g(\phi Y, \phi Z) = g(Y,Z) - \eta(Y)\eta(Z)$

2.1(c). $W_4{}'(\phi X, Y, \phi Z, \xi) = 0$

2.1(d). $W_4{}'(\xi,Y,Z,U) = \eta(U)g(Y,Z) - \eta(Y)g(Z,U)$

2.1(e). $W_4{}'(\xi, \phi Y, Z, \phi U) = 0$

Proof: Set $U = \xi$ in (2.2) we get

$$\begin{aligned} &W_4(X,Y,Z,\xi) \\ &= R'(X,Y,Z,\xi) + \frac{1}{n-1}[g(X,Z)Ric(Y,\xi) - g(X,Y)Ric(Z,\xi)] \\ &= R'(X,Y,Z,\xi) + \frac{1}{n-1}[(\eta-1)\eta(Y)g(X,Z) \\ &\qquad - (\eta-1)\eta(Z)g(X,Y)] \\ &= R'(X,Y,Z,\xi) + \eta(Y)g(X,Z) - \eta(Z)g(X,Y) \\ &\qquad = \{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} + \eta(Y)g(X,Z) \\ &\qquad\qquad - \eta(Z)g(X,Y)\ From\ (1.1) \\ &= g(Y,Z)\eta(X) - \eta(Z)g(X,Y) \end{aligned}$$

Setting $X = U = \xi$ using (2.1)a we get

$$W_4(\nabla, Y, Z, \xi) = g(Y,Z) - \eta(Y)\eta(Z) = g(\phi Y, \phi Z)\ from\ (1.1)$$

Using (1.1) and (2.1) a we obtain

$$W_4(\phi X, Y, \phi Z, \xi) = 0$$

Setting $X = \xi$ in (2.2) and simplifying we get

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$W_4(\xi, Y, Z, U) = \eta(U)g(Y,Z) - g(Z,U)\eta(Y)$ *from* (1.1)

Using (2.1) d and (1.1) we obtain

$W_4(\xi, \phi Y, Z, \phi U) = 0$

Theorem 2: A W_4 –flat K-contact Riemannian manifold is a space of zero Curvature.
Proof: Setting (2.2) to zero we ge

$$W_4{}'(X,Y,Z,U) = R'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Z)Ric(Y,U) - g(X,Y)Ric(Z,U)] = 0$$

which yields

$$R'(X,Y,Z,U) = \frac{1}{n-1}[g(X,Y)Ric(Z,U) - g(X,Z)Ric(Z,U)].$$

Using (1.10) and setting $Y = Z = \xi$ we get

$$R'(X,Y,Z,U) = [g(X,Y)g(Z,U) - g(X,Z)g(Y,U)]$$

$$\Rightarrow R'(X,\xi,\xi,U) = \eta(X)\eta(U) - \eta(X)\eta(U) = 0$$

$\Rightarrow Ric(X,U) = R'(X,\xi,\xi,U) = 0$ and thus follows the theorem.

III. W_4-Symmetric K-contact Riemannian Manifold

A K-contact Riemannian manifold is said to be symmetric if.

(3.1) $\nabla_U W_4(X,Y,Z) = W_4{}'(X,Y,Z,U) = 0$

Theorem 3: A W_4 -symmetric and W_4 -flat K-contact manifold is a flat manifold i.e zero curvature.

Proof: From the symmetric property it follow

(3.1.1) $R(X,Y,W_4(Z,U,V)) - W_4(R(X,Y,Z),U,V) - W_4(Z,R(X,Y,U),V) - W_4(Z,U,R(X,Y,V)) = 0$

We expand the above equation

(3.1.2) $R(X,Y,W_4(Z,U,V)) = g(Y,W_4(Z,U,V))X - g(X,W_4(Z,U,V))Y$

$= W_4{}'(Y,Z,U,V)X - W_4{}'(X,Z,U,V)Y$

$\Rightarrow R'(X,Y,W_4(Z,U,V),\xi) = W_4{}'(Y,Z,U,V)\eta(X) - W_4{}'(X,Z,U,V)\eta(Y)$

(3.1.3) $W_4{}'(R(X,Y,Z),U,V)) = R(R(X,Y,Z),U,V,\xi) + \frac{1}{n-1}[g(R(X,Y,Z)Ric(U,\xi) - g(R(X,Y,Z)Ric(V,\xi)]$

$$\begin{aligned} &= R(R(X,Y,Z),U,V,\xi) \\ &\quad + \frac{n-1}{n-1}[\eta(U)R'(X,Y,Z,V) \\ &\quad - \eta(V)R'(X,Y,Z,V)] \\ &= g(U,V)\eta(R(X,Y,Z)) \\ &\quad - \eta(U)R'(X,Y,Z,V) + \eta(U)R'(X,Y,Z) \\ &\quad - \eta(V)R'(X,Y,Z,U) \\ &= g(U,V)\eta(R(X,Y,Z)) \\ &\quad - \eta(V)R'(X,Y,Z,U) \end{aligned}$$

(3.1.4) $W_4{}'(Z,R(X,Y,U),V,\xi)) = R'(Z,R(X,Y,U),V,\xi)$

$$\begin{aligned} &+ \frac{1}{n-1}[g(Z,V)Ric(R(X,Y,U),\xi) \\ &- g(Z,R(X,Y,U)Ric(V,\xi)] \\ &= R'(Z,R(X,Y,U),V,\xi) \\ &+ \frac{n-1}{n-1}[g(Z,V)\eta(X,Y,U) \\ &- \eta(V)R'(X,Y,U,Z)] \\ &= \eta(Z)R'(X,Y,U,V) \\ &- g(Z,V)\eta(R(X,Y,V)) \\ &+ g(Z,V)\eta(R(X,Y,U)) \\ &- \eta(V)R'(X,Y,U,Z) \\ &= \eta(Z)R'(R(X,Y,U,V)) \\ &- \eta(V)R'(X,Y,U,Z) \end{aligned}$$

(3.1.5) $W_4{}'(Z,U,R(X,Y,V),\xi)) = R'(Z,U,R(X,Y,V),\xi)$

$$\begin{aligned} &+ \frac{1}{n-1}[g(Z,R(X,Y,V))Ric(U,\xi) \\ &- g(Z,U)Ric(R(X,Y,V),\xi] \\ &= R'(Z,U,R(X,Y,V)\xi) \\ &+ \frac{n-1}{n-1}[\eta(U)R'(X,Y,Z) \\ &- g(Z,U)\eta(R(X,Y,Z)) \\ &= R'(X,Y,V,U)\eta(Z) \\ &- \eta(U)R'(X,Y,V,Z) \\ &+ \eta(U)R'(X,Y,V,Z) \\ &- g(Z,U)\eta(R(X,Y,V)) \\ &= R'(X,Y,V)\eta(Z) \\ &- g(Z,U)\eta(R(X,Y,V))] \end{aligned}$$

Putting together (3.1.2), (3.1.3), (3.1.4) and (3.1.5) and simplifying we get

$$\begin{aligned} &W_4{}'(Y,Z,U,V)\eta(X)) \\ &- W_4{}'(X,Z,U,V)\eta(Y)) \\ &= g(U,V)\eta(R(X,Y,Z)) \\ &+ \eta(V)R'(X,Y,Z,U) \\ &- \eta(Z)R'(X,Y,U,V) \\ &+ \eta(V)R'(X,Y,U,Z) \\ &- \eta(Z)R'(X,Y,V,U) \\ &+ g(Z,V)\eta(R(X,Y,V)) = 0 \end{aligned}$$

Terms which are coefficients of $\eta(Z)$ cancel out since R' is skew-symmetric with respect to U and V.

Also terms which are coefficients of $\eta(V)$ cancel out since R' is skew-symmetric with respect to Z and V.

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Thus we remain with

$$
\begin{aligned}
&\mathrm{W}_4{}'(Y,Z,U,V)\eta(X)) \\
&- \mathrm{W}_4{}'(X,Z,U,V)\eta(Y)) \\
&+ g(Z,U)\eta\Big(R(X,Y,U) \\
&- g(U,V)\eta\big(R(X,Y,\xi)\big)\Big) = 0
\end{aligned}
$$

Implying that if W_4 is symmetric i.e $\nabla_X \mathrm{W}_4(X,Y,Z) = \mathrm{W}_4{}'(X,Y,Z,U) = 0$

$$\Rightarrow R\big(R(X,Y,U)\big) = 0$$

Hence the theorem

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