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Strong Law of Large Numbers of Partial Weighted Sums for Pairwise NQD Sequences

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Abstract: **By using the moment inequality, maximal inequality and the truncated method of random variables, we establish the strong law of large numbers of partial weighted sums for pairwise NQD sequences which extend the corresponding result of pairwise NQD random variables.**

Keywords: **Pairwise NQD Sequences, Strong Law of Large Numbers, Truncated Method, Maximal Inequality, Moment Inequality**

I. Introduction

Many known types of negative dependence such as Negatively Associated (NA) and Negatively Orthant Dependence (NOD) etc. have developed on the notion of pairwiseNQD. In (Joag-Dev and Proschan [1]), it was pointed out that an NA sequence is NOD, and gave an example that is NOD but not NA. In particular, among them the Negatively Associated (NA) class is the most important and special case of pairwise NQD class and has wide applications in reliability theory and multivariate statistical analysis. Wang etal. [2] gave an example that is pairwise NQD but not NA. In addition, it is easily seen that an NOD sequence is pairwise NQD from the concept of NOD (see [2]), but the reverse is not true. Thus, pairwise NQD sequences are sequences of wider scope which are weaker than NA and NOD sequences. It is therefore significant to study probabilistic properties of this wider pairwise NQD class. So far, many limiting properties on pairwise NQD sequences have been discussed, for instance, Matula [3] obtained the Kolmogorov strong law of large numbers for pairwise NQD random variable sequences with the same distribution. Wang et al. [4] obtained the Marcinkiewicz weak law of large numbers with the same distribution. Wang et al. [2] obtained the strong stability for Jamison type weighted product sums and the Marcinkiewicz strong law of large numbers for product sums of pairwise NQD sequences. Wu [5] gave the Kolmogorov-type inequality and the three series theorem of pairwise NQD sequences and proved the Marcinkiewicz strong law of large numbers. Chen [6] generalized the results of Matula [3] to the case of non identical distributions under some mild condition. Wan [7] obtained the law of large numbers and complete convergence for pairwise NQD sequences. Gan et al. [8] obtained the strong stability for pairwise NQD sequences. Zhao [9] obtained the almost surely convergence properties and growth rate for partial sums of a class of random variable sequences under moment condition. In addition, Wu [10] obtained the strong convergence rate of mixing sequence based on moment inequality and the truncation method of random variables, and so forth.

Inspired by the papers above, we present the strong law of large numbers for pairwise NQD by using the truncation method below, which extends the corresponding result of pairwise NQD random variables. Put

$$X_i^{(k)} = -\frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}} I\left(X_i < -\frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}\right) + X_i I\left(|X_i|^r \le \frac{2^{k+1}}{(k+1)^{\mu}}\right) + \frac{2^{k+1}}{(k+1)^{\frac{\mu}{r}}} I\left(X_i > \frac{2^{\frac{k+1}{r}}}{(k+1)^{\frac{\mu}{r}}}\right),$$

where I(A) denotes the indicator function of the event A. Denote $S_n = \sum_{i=1}^{n} a\, X_i, S_n^{(k)} = \sum_{i=1}^{n} a\, X_i^{(k)}$.

The symbols C, C_1, C_2, ... stand for generic positive constants not depending on n. $\alpha, \mu\ and\ r$ are positive numbers not depending on n and log x represents $\log_2(\max(x, e))$.

II. Preliminaries and Main Result

Let $X_n, n \ge 1$ be a sequence of random variables defined on a probability space(Ω, F, P). Lehmann [11] introduced the concept of Negatively Quadrant Dependent (NQD) sequences; we have

Definition 1.1. Two random variables X and Y are said to be NQD if for all real numbers x and y, the joint probability density is less than or equal to the product of their marginal probability densities.

i.e. $P(X < x, Y < y) \le P(X < x)P(Y < y).$

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be pairwise NQD if X_i and $X_j$are NQD for any $I, j \in N^+$and $i \ne j$.

To prove the main results, it is necessary to state the following Lemmas;

Lemma 1.1([11]) If random variables X and Y are NQD, then

(i) E(XY) ≤ E(X)E(Y);
(ii)P(X > x, Y > y) ≤ P(X > x)P(Y > y), $\forall x, y \in \mathbb{R}$;

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(iii) If f and g are both non-decreasing (or non-increasing) functions, then f(X) and g(Y) are NQD.

Lemma 1.2 ([5]) Let $\{X_n, n \geq 1\}$ be a pairwise NQD sequence with $E(X_n) = 0$ and $E(X_n^2) < \infty$for all n ≥ 1. Denote $T_j(k) = \sum_{i=j+1}^{j+k} X_i, j \geq 0$. Then

$$E(T_j(k))^2 \leq \sum_{i=j+1}^{j+K} EX_i^2$$

and

$$E(\max_{1 \leq k \leq n} \left(T_j(K)\right)^2) \leq C \log^2 n \sum_{i=j+1}^{j+n} EX_i^2. \qquad (1)$$

Lemma 1.3([12]) Let$\{X_n, n \geq 1\}$ be an arbitrary random variable sequence. If there exists some random variable X such that $P(|X_n| \geq x) \leq CP(|X| \geq x)$ for any x > 0 and n ≥ 1, then for any β> 0 and t > 0,

$$E|X_n|^\beta I(|X_n| \leq t) \leq C\,(E|X|^\beta I(|X| \leq t) + t^\beta P(|X| > t)) \quad (2)$$

and

$$E|X_n|^\beta I(|X_n| > t) \leq CE|X|^\beta I(|X| > t). \qquad (3)$$

Theorem: Let $\{X_n, n \geq 1\}$ be a pairwise NQD sequence with $EX_n = 0$ for all
n≥ 1. Suppose that there exists a random variable X such that for any x> 0 and n ≥ 1,

$$P(|X_n| \geq x) \leq CP(|X| \geq x). \qquad (4)$$

If there exist constants 1≤ r < 2 and α> $(3r/2) - (r+1)$ such that

$$E(|X|^r log^\alpha |X|) < \infty, \qquad (5)$$

then

$$\lim_{n\to\infty} n^{-1/r} S_n = 0, \ a.s. \qquad (6)$$

Proof For any integer n, there exists some integer $k = k(n)$ such that $2^k \leq n < 2^{k+1}$,
hence

$$n^{-1/r}|S_n| \leq \max_{2^k \leq n < 2^{k+1}} (2^{-k/r}|S_n|)$$

It suffices to show that

$$\max_{2^k \leq n < 2^{k+1}} 2^{-k/r}|S_n| \to 0, \qquad a.s., k \to \infty \qquad (7)$$

Take
r < µ < r + 1 and for any ε > 0, denote

$$A_k = \bigcap_{i=1}^{2^{k+1}} (|X_i|^r \leq 2^{k+1}/(k+1)^\mu),$$

$$A_k^c = \bigcup_{i=1}^{2^{k+1}} (|X_i|^r > 2^{k+1}/(k+1)^\mu),$$

$$E_k = (\max_{2^k \leq n < 2^{k+1}} |S_n| > 2^{k/r}\varepsilon).$$

It is clear to check that

$$E_k = E_k A_k + E_k A_k^c$$
$$\subset \left(\max_{2^k \leq n < 2^{k+1}} |S_n^{(k)}| > 2^{k/r}\varepsilon\right) \bigcup \left(\bigcup_{i=1}^{2^{k+1}} (|X_i|^r > 2^{k+1}/(k+1)^\mu)\right).$$

Hence

$$\sum_{k=1}^{\infty} P\left(\max_{2^k \leq n < 2^{k+1}} |s_n| > 2^{k/r}\varepsilon\right) \leq \sum_{k=1}^{\infty} P\left(\bigcup_{i=1}^{2^{k+1}} \left(|X_i|^r > \frac{2^{k+1}}{(k+1)^\mu}\right)\right) + \sum_{k=1}^{\infty} P\left(\max_{2^k \leq n < 2^{k+1}} \left|S_n^{(k)}\right| > 2^{k/r}\varepsilon\right) = I_1 + I_2.$$

If we can obtain that $I_1 < \infty$ and $I_2 < \infty$, by Borel-Cantelli Lemma, expression (7) above holds.
Firstly, we will check $I_1 < \infty$. By inequalities (4), and (5) above, and $1 \leq r < \mu < \alpha$, it follows that

$$I_1 \leq \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k+1}} P\left(|X_i|^r > \frac{2^{k+1}}{(k+1)^\mu}\right) \leq C \sum_{k=1}^{\infty} 2^{k+1} P\left(|X_i|^r > \frac{2^{k+1}}{(k+1)^\mu}\right)$$
$$\leq C_1 \sum_{k=1}^{\infty} 2^{k+1} \sum_{j=k}^{\infty} P\left(\frac{2^j}{j^\mu} \leq |X|^r < \frac{2^{j+1}}{(j+1)\mu}\right)$$
$$= C_1 \sum_{j=1}^{\infty} \sum_{k=1}^{j} 2^{k+1} P\left(\frac{2^j}{j^\mu} \leq |X|^r < \frac{2^{j+1}}{(j+1)\mu}\right)$$
$$= 4C_1 \sum_{j=1}^{\infty} 2^j P\left(\frac{2^j}{j^\mu} \leq |X|^r < \frac{2^{j+1}}{(j+1)\mu}\right)$$
$$\leq C_2 + 4C_1 \sum_{j=j0}^{\infty} \frac{2^j}{j^\mu}(j - \mu \log j)^\alpha E\left(I \frac{2^j}{j^\mu} \leq |X|^r < \frac{2^{j+1}}{(j+1)\mu}\right)$$

(where j_0 satisfies that for $j \geq j_0$, $(j - \mu \log j)^\alpha > 0$ and $1 < \frac{(j-\mu \log j)^\alpha}{j^\mu}$)

$$\leq C_2 + C_3 \sum_{j=j0}^{\infty} E\left\{|X|^r log^\alpha |X| I(\frac{2^j}{j^\mu} \leq |X|^r < \frac{2^{j+1}}{(j+1)\mu})\right\}$$

$$\leq C_2 + C_3 E\left(|X|^r log^{\alpha}|X|\right) < \infty. \quad (8)$$

Next, we will check $I_2<\infty$. By $EX_i= 0$, and expression (3), (4), and (5), and Lemma1.3 and taking k sufficiently large such that $(k + 1 - \mu\, log\,(k + 1))^{\alpha}> 0$, we have

$$\frac{\max_{2^k \leq n < 2^{k+1}} |ES_n^{(k)}|}{2^{k/r}} \leq \sum_{i=1}^{2^{k+1}} \frac{|EX_i^{(k)}|}{2^{k/r}}$$
$$\leq 2^{-k/r}\sum_{i=1}^{2^{k+1}}\left\{E|X_i| I\left(|X_i|^r > \frac{2^{k+1}}{(k+1)^{\mu}}\right) + \frac{2^{(k+1)/r}}{(k+1)^{\mu/r}} P\left(|X_i| > \frac{2^{(k+1)/r}}{(k+1)^{\mu/r}}\right)\right\}$$
$$\leq C2^{-k/r}\sum_{i=1}^{2^{k+1}}\left\{E|X| I\left(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}}\right) + \frac{2^{(k+1)/r}}{(k+1)^{\mu/r}} P\left(|X| > \frac{2^{(k+1)/r}}{(k+1)^{\mu/r}}\right)\right\}$$
$$\leq 2C2^{-k/r}\sum_{i=1}^{2^{k+1}} E|X| I\left(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}}\right)$$
$$\leq C_4 \frac{2^{k+1}(k+1)^{\frac{\mu(r-1)}{r}} E(|X|^r \log^{\alpha}|X|)}{2^{\frac{k}{r}} 2^{(k+1)\frac{(r-1)}{r}} (k+1-\mu\log(k+1))^{\alpha}}$$
$$\leq C_5 \frac{1}{k^{\alpha-\mu+\mu/r}} \to 0 \text{ as } k \to \infty.$$

Hence

$$2^{-k/r} \max_{2^k \leq n < 2^{k+1}} |ES_n^{(k)}| < \frac{\varepsilon}{2} \text{ for k sufficiently large.Thus,}$$

$$I_2 \leq C_6 + \sum_{k=1}^{\infty} P\left(\max_{2^k \leq n < 2^{k+1}} |S_n^{(k)} - ES_n^{(k)}| > 2^{k/r}\varepsilon/2\right). \quad (9)$$

Since $X_i^{(k)} - EX_i^{(k)}$ is a non decreasing function, we have by applying Lemma 1.1(iii) above that $\{X_i^{(k)} - EXik, \leq i \leq n\}$ is still a pairwise NQD sequence with mean zero. Hence byexpression (9) above and, Markov's inequalities, (1), and (2) and C_r inequality, it follows that

$$I_2 \leq C_6 + \sum_{k=1}^{\infty} 2^{-2k/r} E\left(\max_{2^k \leq n < 2^{k+1}} |S_n^{(k)} - ES_n^{(k)}|^2\right)$$
$$\leq C_6 + \sum_{k=1}^{\infty} 2^{-2k/r} E\left(\max_{1 \leq n < 2^{k+1}} \left|\sum_{i=1}^{n}\left(X_i^{(k)} - EX_i^{(k)}\right)\right|^2\right)$$
$$\leq C_6 + \sum_{k=1}^{\infty} \frac{(\log 2^{k+1})^2}{2^{2k/r}} \left\{\sum_{i=1}^{2^{k+1}} E|X_n^{(k)} - EX_n^{(k)}|^2\right\}$$
$$\leq C_6 + C_7 \sum_{k=1}^{\infty} \frac{k^2}{2^{\frac{2k}{r}}} \sum_{i=1}^{2^{k+1}} \left\{E\left(X_i^2 I\left(|X_i|^r \leq \frac{2^{k+1}}{(k+1)^{\mu}}\right)\right) + \frac{2^{\frac{2(k+1)}{r}}}{(k+1)^{\frac{2\mu}{r}}} P(|X_i|^r > \frac{2^{k+1}}{(k+1)^{\mu}})\right\}$$
$$\leq C_6 + C_7 \sum_{k=1}^{\infty} \frac{k^2}{2^{\frac{2k}{r}}} 2^{k+1} E\left(X^2 I(|X|^r \leq \frac{2^{k+1}}{(k+1)^{\mu}})\right)$$
$$\leq C_6 + C_7 \sum_{k=1}^{\infty} \frac{k^2}{2^{\frac{2k}{r}}} 2^{k+1} \frac{2^{\frac{2(k+1)}{r}}}{(k+1)^{\frac{2\mu}{r}}} P\left(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}}\right)\}$$
$$=: C_6 + C_7 I_{21} + C_7 I_{22}.$$

For I_{21}, it is clear to check the fact that $\sum_{n=m}^{\infty} \frac{n(n-1)}{2^{\delta n}} \leq C\frac{m^2}{2^{\delta m}}$ for any$m \geq 1$ and $\delta > 0$.Without loss of generalities, we assume $\frac{2^m}{m^{\mu}} < \frac{2^{m+1}}{(m+1)\mu}, m \geq 1$ and $A_m := \{\frac{2^m}{m^{\mu}} < |X|^r \leq$ *2m+1m+1μ*. Noting that 1 $\leq r<2, r<\mu<r+1, \alpha>(3r/2) - (r+1)$ and $E(|X|^r log^{\alpha}|X|)<\infty$, one has

$$I_{21} = \sum_{k=1}^{\infty} k^2 2^{k+1-\frac{2k}{r}} (EX^2 I|X|^r \leq 2) + \sum_{m=1}^{k} EX^2 I(A_m))$$
$$= C_8 + \sum_{m=1}^{\infty}(\sum_{k=m}^{k} k^2 2^{k+1-2k/r}) EX^2 I(A_m)$$
$$\leq C_8 + C_9 \sum_{m=1}^{\infty} m^2 2^{m-\frac{2m}{r}} E\,|X|^r log^{\alpha}|X|. \frac{|X|^{2-r}}{log^{\alpha}|X|} I(A_m))$$
$$\leq C_8 + C_{10} \sum_{m=1}^{\infty} m^2 2^{m-\frac{2m}{r}} \left(\frac{2^{m-\frac{2m}{r}}}{(m+1)\mu}\right)^{2-r/r} \frac{1}{\left(\log\frac{2^m}{m^{\mu}}\right)^{\alpha}} E|X|^r log^{\alpha}|X| I(A_m)$$
$$\leq C_8 + C_{11} \sum_{m=1}^{\infty} m^{2+\mu-2\mu/r-\alpha} E|X|^r log^{\alpha}|X| I(A_m).$$

Since $\alpha > r$, we can take μ such that $\alpha > 2 + \mu - 2\,\mu/r$. There by

$$I_{21} \leq C_8 + C_{11} E(|X|^r \log|X|) < \infty. \quad (11)$$

$$I_{22} \leq C_6 \sum_{k=1}^{\infty} k^{2-\frac{2\mu}{r}} 2^{k+1} EI(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}})$$
$$\leq C_{12} \sum_{k=1}^{\infty} k^{2+\mu-2\mu/r} E\left(|X|^r I(|X|^r > \frac{2^{k+1}}{(k+1)^{\mu}})\right)$$
$$= C_{12} \sum_{k=1}^{\infty} k^{2-\mu(\frac{2}{r}-1)} \sum_{m=k}^{\infty} E(|X|^r I(A_{m+1})) = C_{12} \sum_{m=k}^{\infty} E(|X|^r I(A_{m+1})) \sum_{k=1}^{m} k^{2+\mu-2\mu/r}$$
$$\leq C_{13} \sum_{m=1}^{\infty} m^{3-\mu(\frac{2}{r}-1)} E(|X|^r I(A_{m+1})) \leq C_{13} \sum_{m=k}^{\infty} m^{(\frac{3r}{2})-(r+2)} E(|X|^r I(A_{m+1}))$$
$$\leq C_{14} + C_{15} \sum_{m=m0}^{\infty} \frac{m^{(\frac{3r}{2})-(r+2)}}{(m+1-\mu\log(m+1))^{\alpha}} E(|X|^r \log^{\alpha}|X| I(A_{m+1}))$$
$$\leq C_{14} + C_{15} \sum_{m=1}^{\infty} E(|X|^r \log^{\alpha}|X| I(A_{m+1}))$$
$$\leq C_{14} + C_{15} E(|X|^r \log^{\alpha}|X|) < \infty. \quad (12)$$

By (10)-(12), we have shown that $I_2<\infty$. Combing (8) with (7), it is seen that expression (6) holds. The proof of the desired result is completed.

Remark In the process of proving $I_2<\infty$, we refer to the method on the proof of Theorem 5.4.2 in [12], but the choices of truncation random variables $\{X_i^{(k)}, 1 \leq i \leq n\}$ and the specific parameter μ are different.

REFERENCES

[1] JOAG-DEV K., PROSCHAN F. Negative association of random variables with applications[J]. The Annals of Statistics, 1983, 11(1): 286-295.

[2] WANG Yuebao, YAN Jigao, CHENG Fengyang, CAI Xinzhong. On the strong stability for Jamison type weighted product sums of pairwise NQD series with different distribution[J]. Chinese Annals of Mathematics, 2001, 22A(6): 701-706.

[3] MATULA P. A note on the almost sure convergence of sums of negatively dependent random variables[J]. Statistics & Probability Letters, 1992, 15(3): 209-213.

[4] WANG Yuebao, SU Chun, LIU Xuguo. On some limit properties for pairwise NQD sequences[J].Acta Mathematicae Applicatae Sinica, 1998, 21(3): 404-414.

[5] WU Qunying. Convergence properties of pairwise NQD random sequences[J]. Acta Mathematica Sinica, 2002, 45(3): 617-624.

[6] CHEN Pingyan. On the strong law of large numbers for pairwise NQD random variables[J]. Acta Mathematica Scientia, 2005, 25A(3): 386-392.

[7] WAN Chenggao. Law of large numbers and complete convergence for pairwise NQD random se-quences[J]. Acta Mathematicae Applicatae Sinica, 2005, 28(2): 253-261.

[8] GAN Shixin, CHEN Pingyan. Strong stability of pairwise NQD random variable sequences[J]. Acta Mathematica Scientia, 2008, 28A(4): 612-618.

[9] ZHAO Ting, HU Shuhe, LI Xiaoqin, et al. Probability inequalities and almost surely convergence properties for a class of random variables[J]. Journal of Anhui University Natural Science Edition,2010, 34(1): 7-10.

[10] WU Qunying. Strong convergence rate for ~_-mixing random sequences with different distributions[J].Journal of Mathematical Research and Exposition, 2004, 24(1): 173-179.

[11] LEHMANN E L. Some concepts of dependence[J]. The Annals of Mathematical Statistics, 1966, 37(5):1137-1153.

[12] WU Qunying. Limit Theorems of Probability Theory for Mixing sequences[M]. Science Press, Beijing,2006, p.173, 229-231.