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Variational Iteration Transform Method for Solving Higher Dimensional Initial Boundary Value Problems

D. Ziane
Laboratory of mathematic and its applications (LAMAP), University of Oran,
P.O. Box 1524, 31000 Oran Algeria.
email : djeloulz@yahoo.com

M. Hamdi Cherif
Laboratory of mathematic and its applications (LAMAP), University of Oran,
P.O. Box 1524, 31000 Oran Algeria.
email : mountassir27@yahoo.fr

***Abstract*: In this paper, the variational iteration transform method(VITM) is employed to obtain approximate analytical solutions of higher dimensional initial boundary value problems . The VITM can easily be applied to many problems and is capable of reducing the size of computational work. The results show that the variational iteration transform method is reliable and efficient to handle linear and nonlinear problems.**

***Keywords*: Variational Iteration Method, Laplace Transform Method, Higher Dimensional Equation, Approximate Analytical Solutions, Linear and Nonlinear Problems.**

I. Introduction

Calculus of variations is an old mathematics, and was originally applied to astronomy by many famous scientists, such as Newton and Jacobi. Due to the remarkable development of computers, many problems can now be solved numerically. As a result the variational approach is rarely used in astronomy and other fields. The variational formulation in energy form has practical physical meanings, and variational approximate solutions are best among all possible trial functions and valid for the whole solution domain [7].

Variational Iteration method was first proposed by He ([1] , [2] , [3] ,[4], [6]). The method gives the solution in the form of a rapidly convergent successive approximations that may give the exact solution if such a solution exists. For concrete problems where exact solution is not obtainable, it was found that a few number of approximations can be used for numerical purposes. The Adomian decomposition method suffers from the cumbersome work needed for the derivation of Adomian polynomials for nonlinear terms. The perturbation method suffers from the computational work specially when the degree of nonlinearity increases. The numerical techniques, such as Galerkin method, also suffer from the need of huge size of computational work. The VIM has no specific requirements for nonlinear operators [9]. Another important advantage is that the VIM is capable of greatly reducing the size of calculations while still maintaining high accuracy of the numerical solution [5].

In recent years, many researchers focused the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform [14]. A particular one is the Laplace homotopy perturbation method by Sweilama and Khader [10], Singh et al. [14], Madani et al. [11], Khan and Wu [12], Kumar et al. [13], and the variational iteration method coupled with Laplace transform method by Kanwal and Mohyud Din [15].

The aim of this paper is to directly apply the variational iteration transform method proposed by Kanwal and Mohyud-Din [15] to consider the rational approximation solution of the higher dimensional initial boundary value problems of variable coefficients.

II. Variational Iteration Method

Consider the differential equation

$$Lu + Nu = g(t), \tag{1}$$

where L and N are linear and nonlinear operators, respectively, and $g(t)$ is the source inhomogeneous term. In ([1] , [2] , [3] ,[4], [6]) the VIM was introduced by He where a correction functional for Eq. (1) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau, \tag{2}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. By this method, it is required first to determine the Lagrangian multiplier that will be identified optimally. The successive approximations $u_{n+1}, n \geq 0$, of the solution u will be readily obtained upon using the determined Lagrangian multiplier and any selective function u_0. Consequently, the solution is given by

$$u = \lim_{n\to\infty} u_n .$$

III. Variational Iteration Method Coupled With Laplace Transform

We consider the general nonlinear, inhomogeneous partial differential equation

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \tag{3}$$

with the initial condition and in this paper L is operator

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$(\frac{\partial^2}{\partial t^2})$

$$u(x,0)=h(x), u_t(x,0)=g(x). \quad (4)$$

Taking the Laplace Transform to the both sides of the given equation

$$\ell Lu(x,t)+\ell Ru(x,t)+\ell Nu(x,t)=\ell f(x,t),$$

with Laplace Transformation

$$s^2\ell Lu(x,t)-su(x,0)-u_t(x,0)=\ell f(x,t)-\ell Ru(x,t)-\ell Nu(x,t).$$

We have

$$\ell Lu(x,t)=\frac{1}{s}h(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\left[\begin{matrix}\ell f(x,t)-\ell Ru(x,t)\\ -\ell Nu(x,t)\end{matrix}\right].$$

Taking the inverse Laplace.

$$u(x,t)=h(x)+g(x)t+\ell^{-1}\left[\frac{1}{s^2}\ell f(x,t)\right]-\ell^{-1}\left[\frac{1}{s^2}\ell Ru(x,t)\right]-\ell^{-1}\left[\frac{1}{s^2}\ell Nu(x,t)\right].$$

Applying $(\frac{\partial}{\partial t})$ on both sides, we have

$$u_t(x,t)-g(x)-\frac{\partial}{\partial t}\ell^{-1}\left[\frac{1}{s^2}\ell f(x,t)\right]+\frac{\partial}{\partial t}\ell^{-1}\left[\frac{1}{s^2}\ell Ru(x,t)\right]+\frac{\partial}{\partial t}\ell^{-1}\left[\frac{1}{s^2}\ell Nu(x,t)\right]=0.$$

The correction functional of the variational iteration method is given as

$$u_{n+1}(t)=u_n(t)-\int_0^t\left[\begin{matrix}\frac{\partial}{\partial\tau}u_n(x,\tau)-g(x)-\\ \frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell f(x,\tau))\\ +\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell Ru_n(x,\tau))\\ +\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell Nu_n(x,\tau))\end{matrix}\right]d\tau,$$

The solution in the series form is given by $u(x,t)=\lim_{n\to\infty}u_n(x,t)$.

IV. Applications

In this section, we apply variational iteration transform method for solving higher dimensional initial boundary value problems with variable coefficient.

Example 1:

We consider the following two-dimensional initial boundary value problem [8]

$$u_{tt}=\frac{1}{2}y^2u_{xx}+\frac{1}{2}x^2u_{yy}, 0<x,y<1, t>0, \quad (5)$$

subject to the boundary conditions

$$\begin{aligned}&u(0,y,t)=y^2e^{-t}, u(1,y,t)=(1+y^2)e^{-t},\\ &u(x,0,t)=x^2e^{-t}, u(x,1,t)=(1+x^2)e^{-t},\end{aligned} \quad (6)$$

and the initial conditions

$$u(x,y,0)=x^2+y^2, u_t(x,y,0)=-(x^2+y^2). \quad (7)$$

Applying the Laplace transform on both sides of Eq. (5), we have

$$\begin{aligned}&s^2\ell u(x,y,t)-su(x,y,0)-u_t(x,y,0)=\\ &\ell\left[\frac{1}{2}(y^2u_{xx}+x^2u_{yy})\right].\end{aligned} \quad (8)$$

Taking the inverse Laplace and applying $(\frac{\partial}{\partial t})$ on both sides of eq. (8), we have

$$u_t(x,y,t)=-(x^2+y^2)+\frac{\partial}{\partial t}\ell^{-1}\left[\frac{1}{s^2}\ell(\frac{1}{2}(y^2u_{xx}+x^2u_{yy}))\right]. \quad (9)$$

The correction functional of the variational iteration method is given as

$$u_{n+1}(t)=u_n(t)-\int_0^t\left[\begin{matrix}\frac{\partial}{\partial\tau}u_n-\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell(\frac{1}{2}(y^2u_{nxx}+x^2u_{nyy})))\\ +(x^2+y^2)\end{matrix}\right]d\tau. \quad (10)$$

This in turn gives the successive approximations

$$u_0(x,y,t)=(x^2+y^2)(1-t),$$

$$u_1(x,y,t)=(x^2+y^2)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}),$$

$$u_2(x,y,t)=(x^2+y^2)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}), \quad (11)$$

$$u_3(x,y,t)=(x^2+y^2)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}-\frac{t^7}{7!}),$$
$$\vdots$$
$$u_n(x,y,t)=(x^2+y^2)(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}-\frac{t^7}{7!}+...+\frac{(-t)^n}{n!}).$$

Recall that the exact solution is given by

$$u(x,y,t)=\lim_{n\to\infty}u_n(x,y,t).$$

This in turn gives the exact solution

$$u(x,y,t)=(x^2+y^2)e^{-t}, \qquad (12)$$

which is an exact solution to the equation (5) as presented in [8].

Example 2:

No, we consider the three-dimensional initial boundary value problem [8]

$$u_{tt}=\frac{1}{45}x^2u_{xx}+\frac{1}{45}y^2u_{yy}+\frac{1}{45}z^2u_{zz}-u, 0<x,y,z<1, t>0, \qquad (13)$$

subject to the boundary conditions

$$\begin{aligned}&u_x(0,y,z,t)=0, u_x(1,y,z,t)=6y^6z^6\sinh t,\\&u_y(x,0,z,t)=0, u_y(x,1,z,t)=6x^6z^6\sinh t,\\&u_z(x,y,0,t)=0, u_z(x,y,1,t)=6x^6y^6\sinh t,\end{aligned} \qquad (14)$$

and the initial conditions

$$u(x,y,z,0)=0, u_t(x,y,z,0)=x^6y^6z^6. \qquad (15)$$

In a similar way as above we have

$$u_{n+1}(t)=u_n(t)-\int_0^t\left[\begin{array}{l}\frac{\partial}{\partial\tau}u_n-x^6y^6z^6\\-\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell(\frac{1}{45}x^2u_{nxx}\\+\frac{1}{45}y^2u_{nyy}+\frac{1}{45}z^2u_{nzz}-u_n))\end{array}\right]d\tau. \qquad (16)$$

Using the initial condition (15) and the iteration formula (16) we obtain the following approximations

$$u_0(x,y,z,t)=x^6y^6z^6t,$$

$$u_1(x,y,z,t)=x^6y^6z^6(t+\frac{t^3}{3!}),$$

$$u_2(x,y,z,t)=x^6y^6z^6(t+\frac{t^3}{3!}+\frac{t^5}{5!}),$$

$$u_3(x,y,z,t)=x^6y^6z^6(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}),$$
$$\vdots \qquad (17)$$
$$u_n(x,y,z,t)=x^6y^6z^6(t+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+...+\frac{t^{2n+1}}{(2n+1)!}).$$

Recall that the exact solution is given by

$$u(x,y,z,t)=\lim_{n\to\infty}u_n(x,y,z,t).$$

This in turn gives the exact solution

$$u(x,y,z,t)=x^6y^6z^6\sinh t. \qquad (18)$$

which is an exact solution to the equation (13) as presented in [8].

Example 3:

We consider the following two-dimensional nonlinear inhomogeneous initial boundary value problem [8]

$$u_{tt}=2(x^2+y^2)+\frac{15}{2}(xu^2{}_{xx}+yu^2{}_{yy}), 0<x,y<1, t>0, \qquad (19)$$

with boundary conditions

$$\begin{aligned}&u(0,y,t)=y^2t^2+yt^6, u(1,y,t)=(1+y^2)t^2+(1+y)t^6,\\&u(x,0,t)=x^2t^2+xt^6, u(x,1,t)=(1+x^2)t^2+(1+x)t^6,\end{aligned} \qquad (20)$$

and the initial conditions

$$u(x,y,0)=0, u_t(x,y,0)=0. \qquad (21)$$

In a similar way as above we have

$$u_{n+1}=u_n-\int_0^t\left[\begin{array}{l}\frac{\partial}{\partial\tau}u_n-\\\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell(\frac{15}{2}(xu^2{}_{nxx}+yu^2{}_{nyy})))\\-\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell(2(x^2+y^2)))\end{array}\right]d\tau. \qquad (22)$$

Using the initial condition (21) and the iteration formula (22) we obtain the following approximations

$$u_0(x,y,t)=0,$$

$$u_1(x,y,t)=(x^2+y^2)t^2,$$

$$u_2(x,y,t)=(x^2+y^2)t^2+(x+y)t^6,$$

$$u_3(x,y,t)=(x^2+y^2)t^2+(x+y)t^6,$$
$$\vdots \quad (23)$$
$$u_n(x,y,t)=(x^2+y^2)t^2+(x+y)t^6.$$

This in turn gives the exact solution

$$u(x,y,t)=(x^2+y^2)t^2+(x+y)t^6. \quad (24)$$

which is an exact solution to the equation (19) as presented in [8].

Example 4:

Finally, we consider the three-dimensional nonlinear initial boundary value problem [8]

$$u_{tt}=(2-t^2)+u-(e^{-x}u^2{}_{xx}+e^{-y}u^2{}_{yy}+e^{-z}u^2{}_{zz}), \quad (25)$$

where $0<x,y,z<1, 0<t\le 1$, with boundary conditions

$$u_x(0,y,z,t)=1, u_x(1,y,z,t)=e,$$
$$u_y(x,0,z,t)=1, u_y(x,1,z,t)=e,$$
$$u_z(x,y,0,t)=1, u_z(x,y,1,t)=e, \quad (26)$$

and the initial conditions

$$u(x,y,z,0)=e^x+e^y+e^z, u_t(x,y,z,0)=0. \quad (27)$$

In a similar way as above we have

$$u_{n+1}=u_n-\int_0^t \begin{bmatrix} \frac{\partial}{\partial\tau}u_n+ \\ \frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell(e^{-x}u^2{}_{nxx}+ \\ e^{-y}u^2{}_{nyy}+e^{-z}u^2{}_{nzz}-u_n)) \\ -\frac{\partial}{\partial\tau}\ell^{-1}(\frac{1}{s^2}\ell(2-\tau^2)) \end{bmatrix} d\tau. \quad (28)$$

Using the initial condition (27) and the iteration formula (28) we obtain the following approximations

$$u_0(x,y,z,t)=e^x+e^y+e^z,$$

$$u_1(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^4}{12},$$

$$u_2(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^6}{360},$$

$$u_3(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^8}{20160},$$

$$u_4(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^{10}}{1814400}, \quad (29)$$

$$u_5(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^{12}}{239500800},$$

$$u_6(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^{14}}{4358914560\,0},$$

$$u_7(x,y,z,t)=e^x+e^y+e^z+t^2-\frac{t^{16}}{3923023104\,0000},$$

$$\vdots$$

This in turn gives the exact solution

$$u(x,y,z,t)=e^x+e^y+e^z+t^2, \quad (30)$$

which is an exact solution to the equation (25) as presented in [8].

V. Conclusion

In this paper, we have applied the variational iteration transform method (VITM) for solving higher dimensional initial boundary value problems with variable coefficients. From the results, it is clear that the variational iteration transform method yields very accurate approximate solutions using only a few iterates. Thus, it is concluded that the VITM becomes more powerful and efficient than before in finding analytical, as well as numerical, solutions for a wide class of nonlinear differential equations. It provides more realistic series solutions that converge very rapidly in real physical problems.

The fact that the variational iteration transform method solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this algorithm over the decomposition method.

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AUTHOR'S PROFILE

Djelloul Ziane
email: djeloulz@yahoo.com
Born in Chlef, Algeria on 1967. Member of Laboratory of mathematic and its applications (LAMAP), University of Oran, Algeria. Professor in the department of Mathematics, University of Chlef, Algeria.

Mountassir Hamdi Cherif
email: mountassir27@yahoo.fr
Born in Achaacha (Mostaganem), Algeria on 1988. Member of Laboratory of mathematic and its applications (LAMAP), University of Oran, Algeria. Teaching in the preparatory school in science and technology, Oran, Algeria.